The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

Numbers and Groups **2 hours**

Answer ***all questions****.*
You should justify your answers carefully unless the question states otherwise.

1 (i) Suppose that $7x \equiv 4 \pmod{19}$. Prove that $x \equiv 6 \pmod{19}$. ***(3 marks)***

(ii) State the Chinese Remainder Theorem. ***(2 marks)***

(iii) Suppose now that $7x \equiv 4 \pmod{19}$ and also $x \equiv 9 \pmod{14}$. Express everything we know about x in terms of a single congruence of the form $x \equiv a \pmod{m}$. Explain all your working. ***(5 marks)***

2 (i) Given a sequence $a_1, a_2, \ldots$, state what it means for the sequence to be *convergent*. ***(2 marks)***

(ii) For each of the following sequences, say whether they converge or not. Give brief justification for your answers, and give the limit (where it exists).

(a) $a_n = \pi$; ***(2 marks)***

(b) $b_n = (-1)^n$; ***(2 marks)***

(c) $c_n = (-\pi)^n$; ***(2 marks)***

(d) $d_n = \left(-\frac{1}{\pi}\right)^n$. ***(2 marks)***

3 Recall that the *factorial* $n!$ is defined to be $1 \times 2 \times \cdots \times (n-1) \times n$.

(i) Prove by induction that $n! > 3^n$ for all $n \geq 7$. ***(4 marks)***

(ii) Find all positive integers n for which $n! < 3^n$. ***(2 marks)***

(iii) In contrast, prove that $n! < 3^{n^2}$ for all $n \geq 1$. *[Remember that a^{b^c} means $a^{(b^c)}$.]* ***(4 marks)***

4 As usual, let S_n denote the group of permutations of the set $\{1, \ldots, n\}$.

(i) (a) In S_{11}, find the cycle decompositions of

$$\begin{aligned}\alpha &= (1\ 7\ 8\ 9\ 10\ 11)(1\ 2\ 3\ 4\ 5\ 6);\\ \beta &= (2\ 1\ 7\ 8\ 9\ 10)(1\ 2\ 3\ 4\ 5\ 6);\\ \gamma &= (3\ 2\ 1\ 7\ 8\ 9)(1\ 2\ 3\ 4\ 5\ 6).\end{aligned}$$

(3 marks)

(b) Find a cycle $(a\ b\ c\ d\ e\ f)$ of length 6 such that $(a\ b\ c\ d\ e\ f)(1\ 2\ 3\ 4\ 5\ 6)$ is a cycle of length 5. ***(1 mark)***

(c) Is it possible to find two six-cycles whose composition is a cycle of even length? Justify your answer. ***(1 mark)***

(ii) Let $n \geq 3$. For each of the subsets of S_n below, determine whether or not it is a subgroup of S_n, justifying your answers.

$$\begin{aligned}H_1 &= \{\alpha \in S_n : \alpha(1) = 2\};\\ H_2 &= \{\alpha \in S_n : \alpha(1) = 1\};\\ H_3 &= \{\alpha \in S_n : \alpha^2 = \mathrm{id}\}.\end{aligned}$$

(5 marks)

5 Let $(G, \odot)$ be a group of order 15, and let H be a subgroup of G. For each statement below, determine whether it is true or false. (Note: you must justify your answers to get the marks.)

(i) If $a, b \in G$ then $a \odot b \in G$.

(ii) There is an element $e \in G$ such that $e \odot g = g = g \odot e$ for all $g \in G$.

(iii) If $a, b, c \in G$ with $ab = ac$, then $b = c$.

(iv) If $a, b, c \in G$ with $ba = ca$, then $b = c$.

(v) H is a subset of G.

(vi) H must have order 1, 3, 5 or 15.

(vii) If H has order 5 then there are 3 left-cosets of H in G.

(viii) If H has order 3 then every left-coset of H in G contains 5 elements.

(ix) The relation on G given by $aRb \iff b^{-1}a \in H$ is symmetric.

(x) G must not be abelian.

(10 marks)

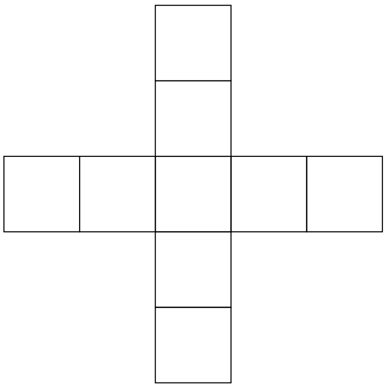
6 (i) (a) Let G be a group acting on a non-empty set X. For $x \in X$, define stab(x) and orb(x). ***(2 marks)***

(b) There is an action of the symmetric group S_3 on polynomials in x_1, x_2 and x_3, whereby a permutation α sends x_i to $x_{\alpha(i)}$ (for $i = 1, 2, 3$). For example,

$$(1\ 2\ 3) * (x_1 + x_2^2 - x_3) = x_2 + x_3^2 - x_1.$$

Let $p = x_1^2 + x_2 + x_3$. For the action described above, list all of the elements of stab(p) and orb(p). ***(2 marks)***

(ii) An ornament, which can be turned-over, is to be made by gluing together 9 coloured plastic squares to make the shape below.



Find the number of essentially different ways this can be done when there is an unlimited supply of pieces in each of n colours, making your workings clear. ***(6 marks)***

End of Question Paper