The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014–15**

Topics in Advanced Fluid Mechanics **2 hours 30 minutes**

Marks will be awarded for your best ***four*** *answers.*

1 Consider the Euler equations for an incompressible fluid. The vorticity $\boldsymbol{\omega}$ and the impulse $\boldsymbol{\gamma}$ (in the geometric gauge) satisfy

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u},$$

$$\frac{D\boldsymbol{\gamma}}{Dt} = -(\nabla \boldsymbol{u})^T \boldsymbol{\gamma},$$

where the velocity $\boldsymbol{u}$ satisfies $\nabla \cdot \boldsymbol{u} = 0$ and T denotes matrix transpose. Throughout this question we assume $i, j = 1, 2, 3$ and summation convention is assumed on repeated indices.

(i) Derive the equations for the Jacobian matrix as

$$\frac{D}{Dt}\frac{\partial x_i}{\partial a_j} = \frac{\partial u_i}{\partial x_k}\frac{\partial x_k}{\partial a_j}.$$

Here $\boldsymbol{x}(\boldsymbol{a}, t)$ denotes the location of a fluid particle $\boldsymbol{a}$ at time t, with the assumption $\boldsymbol{x} = \boldsymbol{a}$ at $t = 0$. ***(5 marks)***

(ii) We confirm a first integral for the Euler equations using the equation for $\boldsymbol{\omega}$. To do so, assuming the form $\omega_i(\boldsymbol{a}, t) = C_j(\boldsymbol{a}, t)\dfrac{\partial x_i}{\partial a_j}$, derive

$$\frac{DC_i}{Dt} = 0.$$

Hence deduce that

$$\omega_i(\boldsymbol{a}, t) = \omega_j(\boldsymbol{a}, 0)\frac{\partial x_i}{\partial a_j}.$$

(8 marks)

(iii) We confirm a first integral for the Euler equations using the equation for $\boldsymbol{\gamma}$. Assuming the form $\gamma_i(\boldsymbol{a}, t) = B_j(\boldsymbol{a}, t)\dfrac{\partial a_j}{\partial x_i}$, derive

$$\frac{DB_i}{Dt} = 0.$$

Hence deduce that

$$\gamma_i(\boldsymbol{a}, t) = \gamma_j(\boldsymbol{a}, t)\frac{\partial a_j}{\partial x_i}.$$

Hint: you may use the equations for the inverse Jacobian matrix

$$\frac{D}{Dt}\frac{\partial a_j}{\partial x_i} = -\frac{\partial a_j}{\partial x_k}\frac{\partial u_k}{\partial x_i}.$$

(8 marks)

(iv) Show that $\dfrac{D}{Dt}(\boldsymbol{\gamma} \cdot \boldsymbol{\omega}) = 0$. ***(4 marks)***

2 Consider the three-dimensional Burgers equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \triangle \boldsymbol{u} = 0.$$

(i) By assuming $\boldsymbol{u} = \nabla \phi$, show that the left-hand side can be written as

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \triangle \boldsymbol{u} = \nabla \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 - \nu \triangle \phi \right).$$

(10 marks)

(ii) Prove an identity

$$\triangle \log \psi = \frac{\triangle \psi}{\psi} - \frac{|\nabla \psi|^2}{\psi^2},$$

which holds for *any* smooth function ψ. ***(5 marks)***

(iii) By taking $\phi = -2\nu \log \psi$, show that

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \triangle \boldsymbol{u} = -2\nu \nabla \left(\frac{\frac{\partial \psi}{\partial t} - \nu \triangle \psi}{\psi} \right).$$

(10 marks)

3 Consider a model equation for the vorticity defined in $\mathbb{R}^1$:

$$\frac{\partial \omega}{\partial t} + U\frac{\partial \omega}{\partial x} = \omega H[\omega], \tag{1}$$

with an initial condition $\omega(x, t = 0) = \omega_0(x)$. Here U is a constant independent of x and t, and $H[\omega]$ denotes the Hilbert transform

$$H[\omega](x) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{\omega(y)}{x - y} dy,$$

defined with a principal-value integral.

(i) Show that $\omega = F(x - Ut)$ is a solution to the linearised version of (1), that is,

$$\frac{\partial \omega}{\partial t} + U\frac{\partial \omega}{\partial x} = 0,$$

where $F(x) = \omega_0(x)$. ***(5 marks)***

(ii) By a set of transformations

$$\begin{cases} X = x - Ut, \\ T = t, \end{cases}$$

derive from (1) an equation for $\omega = \Omega(X, T)$

$$\frac{\partial \Omega}{\partial T} = \Omega H[\Omega]. \tag{2}$$

Here $H[\Omega]$ denotes the Hilbert transform with respect to X. ***(8 marks)***

(iii) Consider

$$\Omega(X, T) = \frac{X}{X^2 + a(T)^2}$$

and determine $a(T)$ such that it is a solution to (2).

Hint: You may use

$$H\left[\frac{a}{X^2 + a^2}\right] = \frac{X}{X^2 + a^2}, \quad H\left[\frac{x}{X^2 + a^2}\right] = -\frac{a}{X^2 + a^2},$$

which hold for any constant a. ***(7 marks)***

(iv) In the case $a(0) < 0$, determine the position x and the time t at which $\omega(x, t)$ becomes singular, that is, unbounded. ***(5 marks)***

4 We consider a system of N point-vortices subject to a time-independent flow:

$$\frac{dx_i}{dt} = -\frac{1}{2\pi}\sum_{j=1}^{N}{}' \kappa_j \frac{y_i - y_j}{(x_i - x_j)^2 + (y_i - y_j)^2} + ax_i - by_i,$$

$$\frac{dy_i}{dt} = \frac{1}{2\pi}\sum_{j=1}^{N}{}' \kappa_j \frac{x_i - x_j}{(x_i - x_j)^2 + (y_i - y_j)^2} + bx_i - ay_i,$$

where $\sum'$ denotes a summation excluding $j = i$. Here (x_i, y_i), $i = 1, 2, \ldots, N$ denotes coordinates of a point vortex of strength κ_i and a, b are constants.

(i) For $a = b = 0$, show that

$$H = -\frac{1}{8\pi}\sum_{j,k=1}^{N}{}' \kappa_j \kappa_k \log\left((x_j - x_k)^2 + (y_j - y_k)^2\right)$$

is a constant of motion, by rewriting the above set of equations in a canonical form

$$\kappa_j \frac{dx_j}{dt} = \frac{\partial H}{\partial y_j}, \quad \kappa_j \frac{dy_j}{dt} = -\frac{\partial H}{\partial x_j}, \text{ (no summation).}$$

(7 marks)

(ii) For $a = b = 0$, show that

$$\sum_{i=1}^{N} \kappa_i x_i, \ \sum_{i=1}^{N} \kappa_i y_i, \ \sum_{i=1}^{N} \kappa_i (x_i^2 + y_i^2)$$

are constants of motion. ***(10 marks)***

(iii) By adding extra terms to the Hamiltonian H suitably, put the system for $(a, b) \neq (0, 0)$ in a canonical form. On this basis, give a reason why the three quantities in (ii) are *no longer* constants of motion. ***(8 marks)***

5 Consider a vortex patch with unit vorticity in a domain D whose boundary is ∂D. The symbols $\boldsymbol{x} = (x, y)$ and $\boldsymbol{x'} = (x', y')$ denote spatial positions on $\mathbb{R}^2$.

(i) Derive the dynamical equation for the vortex patch

$$\frac{d\boldsymbol{x}}{dt} = -\frac{1}{2\pi}\int_{\partial D} \log|\boldsymbol{x} - \boldsymbol{x}'| d\boldsymbol{x}',$$

by making use of the following hints. Here $d\boldsymbol{x}' \equiv (dx', dy')^T$ denotes a line element.

Hint: localised vorticity ω is related to the stream function ψ by

$$\psi(\boldsymbol{x}) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \omega(\boldsymbol{x}') \log|\boldsymbol{x} - \boldsymbol{x}'| dx'dy'.$$

Hint: Green's Theorem states that

$$\int_{\partial D} f dx + g dy = \int_D \left(-\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x}\right) dx dy.$$

(10 marks)

(ii) Recast the dynamical equation for a vortex patch in complex notation as

$$\frac{\partial z(\alpha, t)}{\partial t} = -\frac{1}{2\pi}\int_0^{2\pi} \log|z(\alpha, t) - z(\beta, t)| \frac{\partial z(\beta, t)}{\partial \beta} d\beta.$$

Here $z(\alpha, t) = x(\alpha, t) + iy(\alpha, t)$ denotes the position of the boundary of the patch with Lagrangian marker variable α: $0 \le \alpha \le 2\pi$. ***(4 marks)***

(iii) Show that $z(\alpha, t) = \exp(it/2 + i\alpha)$ is a solution to the above equation. Give a physical meaning of this solution. ***(11 marks)***

End of Question Paper