The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014–15**

Methods for Differential Equations **2 hours**

Answer all four questions

Please leave this exam paper on your desk
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to be completed by student

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1 (i) The equations governing the two competing species x and y are

$$\dot{x} = x(1 - 2x - y), \;\; \dot{y} = 2y(1 - y - 2xy),$$

where $x(\geq 0)$ and $y(\geq 0)$ are measured in appropriate units. Find the equilibrium points.

(3 marks)

(ii) If $V(x, y) = px^2 + qy^4$ where p and q are constants with $p > 0$ and $q > 0$, show that $V(x, y)$ can be a strong Liapunov function for the system

$$\dot{x} = -2xy^4 - x^5, \;\; \dot{y} = x^2y - y^3 \; ,$$

for the equilibrium point (0,0).

State what can be deduced about the nature of this equilibrium point.

(9 marks)

(iii) Transform $y'' - \dfrac{2m}{x}y' + \left[1 + \dfrac{m(m+1)}{x^2}\right] y = 0$ into normal form, and hence find its general solution.

(13 marks)

2 (i) Find the linear approximation to the following system in the neighbourhood of the point (0,0), and determine whether the point (0,0) is stable, asymptotically stable or unstable.

$$\dot{x} = -x - y - 3x^2y, \qquad \dot{y} = -x - 4y$$

(5 marks)

(ii) Are the following quadratic functions positive definite, negative definite, or neither?

$$(a)\ x^2 - xy - y^2, \quad (b)\ 2x^2 - 3xy + 3y^2.$$

(6 marks)

(iii) Let

$$V = \alpha x^2 + \beta x^4 + y^2\ ,$$

where α and β are positive constants, and

$$\dot{x} = y, \quad \dot{y} = -cy - x - x^3\ ,$$

where c is a positive constant. Find suitable values of α and β, for which V is a weak Liapunov function, but not a strong Liapunov function, for the equilibrium point $(0,0)$.

State what can be deduced about the nature of this equilibrium point.

(14 marks)

3 (i) Find all the eigenvalues λ_n $(n = 1, 2, \ldots)$ with $\lambda_1 < \lambda_2 < \ldots$ of the eigenvalue problem

$$y'' - 2y' + \lambda y = 0$$

$$y(0) = 0, \qquad y(1) = 0 \ ,$$

and show that the corresponding eigenfunctions are $y_n(x) = B_n e^x \sin \alpha_n x$, where $B_n \, (n = 1, 2, ...)$ are constants and α_n are the positive roots of $\sin \sqrt{\lambda - 1} = 0$.

You should consider separately the cases:

(a) $\lambda < 1$; (b) $\lambda > 1$.

(14 marks)

(ii) Show that $x = 0$ is a regular singular point of the differential equation

$$4x^2 y'' - 2xy' + (2 + x)y = 0 \ .$$

(5 marks)

Using the Frobenius expansion:

$$y = \sum_{n=0}^{\infty} a_n x^{n+\alpha} \ , \qquad a_0 \neq 0 \ ,$$

show that the roots of the indicial equation are 1 and $\frac{1}{2}$.

(6 marks)

4 (i) By means of the substitution $y = x^{\frac{1}{2}} z$, show that

$$y'' + x^2 y = 0 \qquad (*)$$

transforms into

$$x^2 z'' + x z' + (x^4 - \frac{1}{4}) z = 0 \ . \qquad (**)$$

(5 marks)

If $t = \dfrac{x^2}{2}$, show that (**) can be re-written in the form

$$t^2 \frac{d^2 z}{dt^2} + t \frac{dz}{dt} + \left(t^2 - \frac{1}{16}\right) z = 0 \ ,$$

which is Bessel's equation of order $\nu = \dfrac{1}{4}$. ***(7 marks)***

Hence write down the general solution of $(*)$ in terms of Bessel functions.
(3 marks)

(ii) Find power series solution about $x = 0$ for the following second order ordinary differential equation.

$$\frac{d^2 y}{dx^2} - x^2 \frac{dy}{dx} + xy = 0$$

(10 marks)

End of Question Paper