The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014–15**

Mathematics Core 1 **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 Let $A := \{1, 2, 3\}$ and $B := \{1, 4, 5\}$.

(i) Determine $|A \cup B|$, $|A \cap B|$ and $|A \setminus B|$, and write down three distinct elements of $(A \setminus B) \times B$. ***(4 marks)***

(ii) Write down two different functions from A to B. How many different functions from A to B are there altogether? You do not have to justify your answer. ***(2 marks)***

2 Prove by using induction that $1^2 + 3^2 + \ldots + (2n-1)^2 = \dfrac{4n^3 - n}{3}$ for all $n \in \mathbb{N}$. ***(5 marks)***

3 Let $f(x) := \dfrac{1}{x^2}$ with domain $\{x \in \mathbb{R} : x > 0\}$.

(i) Write down $\dfrac{df}{dx}$. ***(1 mark)***

(ii) By considering $f(x+h) - f(x)$ for small $h \neq 0$, obtain the formula you gave in part (i). ***(4 marks)***

4 Let Γ be the solid obtained by revolving about the x-axis the part of the curve $y = \sqrt{1 + x^3}$ between $x = 0$ and $x = 1$. Write down an expression for dV, the volume of an infinitesimally thin slice of the solid Γ at x of thickness dx, and find the volume of Γ. ***(4 marks)***

5 Let $z_1 := 4 + 3i$ and $z_2 := 12 + 5i$.

(i) Express $z_1 z_2$ in the form $a + bi$, $a, b \in \mathbb{R}$. Also find $|z_1|$ and $|z_2|$.
(2 marks)

(ii) Find a complex number z such that $z^2 = iz_1$. ***(2 marks)***

(iii) Find a right angled triangle whose side lengths are integers and hypotenuse has length 65. Find a second non-congruent right angled triangle with the same property. In both cases it is sufficient to write down the side lengths.
(4 marks)

6 State Euler's identity for $e^{i\theta}$. Prove that

$$\left(\frac{1 + i\tan\theta}{1 - i\tan\theta}\right)^a = \frac{1 + i\tan a\theta}{1 - i\tan a\theta}$$

for all $a \in \mathbb{R}$. You may assume the complex exponential satisfies the usual rules of exponentiation. You can also ignore the cases when θ and $a\theta$ are odd integral multiples of $\frac{\pi}{2}$. ***(4 marks)***

7 Write down the Maclaurin series expansion for $\frac{1}{1+x}$ and use it to show that

$$\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots.$$

Hence or otherwise derive the Maclaurin series expansion for $\ln\left(\frac{1+x}{1-x}\right)$. You do not need to worry about the radii of convergence of the series involved.
(6 marks)

8 (i) Define $\cosh x$ and $\sinh x$ in terms of the exponential function. Derive formulae for the derivatives of these functions, and also show that $\cosh^2 x - \sinh^2 x = 1$. ***(4 marks)***

(ii) Let $y = \sinh^{-1}(x)$, so $\sinh y = x$. Express $\cosh y$ in terms of x and show that $y = \ln(x + \sqrt{1 + x^2})$. Also, express the derivative $\frac{dy}{dx}$ in terms of x.
(5 marks)

9 In this question y is a function of t. Thus $y' = \dfrac{dy}{dt}$ and $y'' = \dfrac{d^2y}{dt^2}$. You should assume that the variable is always positive i.e. $t > 0$.

(i) Find the general solution to the differential equation $y'' - 3y' + 2y = 0$.
(2 marks)

(ii) Find the general solution to the differential equation $y'' - 3y' + 2y = e^t$.
(3 marks)

(iii) Find the general solution to the differential equation $t^2y' + y^2 = ty$. You might want to try the substitution $y = ut$. ***(4 marks)***

10 (i) We want to divide 12 people into 3 labelled teams distinguished by the colour of hats they wear: 4 people will wear red hats, 4 people will wear blue hats and 4 people will wear green hats. In how many ways can this be achieved? There is no need to justify your answer and you may leave your answer as a factorial. ***(1 mark)***

(ii) In how many ways can you divide 12 people into 3 unlabelled groups of 4 people each? Again, you may leave your answer as a factorial and you do not need to give a justification. ***(1 mark)***

(iii) Let m, n be positive integers. Show that

$$\frac{(mn)!}{(m!)^n n!}$$

is an integer by finding an interpretation of the above fraction as the solution to some counting problem. You do not have to justify why the fraction counts what you claim. ***(2 marks)***

End of Question Paper