The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

MAS451 Measure and Probability **2 hours**

Full marks may be obtained by complete answers to three questions. All answers will be marked, but credit will be given only for the best three answers. Total marks 99.

1 (i) Let S be a set. Give precise definitions of

(a) A *outer measure* μ^* on S. ***(4 marks)***

(b) The *Lebesgue outer measure* λ^* on the real number line. ***(2 marks)***

(ii) If μ^* is an outer measure on S deduce that

$$\mu^*(A) \leq \mu^*(A \cap B) + \mu^*(A \cap B^c),$$

for all $A, B \subseteq S$. ***(3 marks)***

(iii) Let $\mathcal{M}_{\mu^*}(S)$ denote the collection of all subsets B of S for which

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c),$$

for all $A \subseteq S$. If (B_n) is a sequence of mutually disjoint sets in $\mathcal{M}_{\mu^*}(S)$, use induction to show that for all $A \subseteq S$ and all $n \in \mathbb{N}$,

$$\mu^*(A) = \sum_{i=1}^{n} \mu^*(A \cap B_i) + \mu^*\left(A \cap \bigcap_{i=1}^{n} B_i^c\right).$$

(7 marks)

(iv) Deduce that $\mathcal{M}_{\mu^*}(S)$ is a σ-algebra. [Hint: To show that $\mathcal{M}_{\mu^*}(S)$ is closed under countable unions, it's sufficient to take the sets in the union to be mutually disjoint.] ***(8 marks)***

(v) Let $A \subseteq \mathbb{R}$.

(a) For each $N \in \mathbb{N}$, deduce that there is an open set O_N in $\mathbb{R}$ such that $A \subseteq O_N$ and

$$\lambda(O_N) \leq \lambda^*(A) + \frac{1}{N}.$$

(3 marks)

(b) Show that $B = \bigcap_{N=1}^{\infty} O_N$ is a Borel set in $\mathbb{R}$ for which

$$\lambda^*(A) = \lambda(B).$$

(6 marks)

2 Throughout this question (S, Σ, m) is a measure space and $\mathbb{R}$ is equipped with its usual Borel σ-algebra.

(i) Recall that $f : S \to \mathbb{R}$ is a measurable function if $f^{-1}((a, \infty)) \in \Sigma$ for all $a \in \mathbb{R}$. Show that this is equivalent to requiring $f^{-1}([a, \infty)) \in \Sigma$ for all $a \in \mathbb{R}$. ***(4 marks)***

(ii) (a) Let f and g be measurable functions defined on S and define

$$(f \wedge g)(x) = \min\{f(x), g(x)\} \text{ for all } x \in \mathbb{R}.$$

Show that $f \wedge g$ is measurable. ***(2 marks)***

(b) If $f_1, f_2, \ldots, f_n$ are measurable functions on $\mathbb{R}$, deduce that $f_1 \wedge f_2 \wedge \cdots \wedge f_n$ is measurable. ***(2 marks)***

(iii) (a) Suppose that g and h are measurable functions on S and $A \in \Sigma$. For each $x \in S$, define $f(x) = \begin{cases} g(x) & \text{if } x \in A \\ h(x) & \text{if } x \notin A \end{cases}$.

Is f measurable? Justify your answer. ***(5 marks)***

(b) Let (f_n) be a sequence of measurable functions defined on S and (A_n) be a sequence of mutually disjoint sets where $A_n \in \Sigma$ for all $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} A_n = S$. Define

$$f(x) = f_n(x) \text{ if } x \in A_n.$$

Is f measurable? Justify your answer. ***(5 marks)***

(iv) Suppose that (f_n) is a sequence of measurable functions defined on S and converging pointwise almost everywhere to a measurable function f, so that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in S - A$ where $A \in \Sigma$ with $m(A) = 0$. Let h be a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Define the functions $G_n = h \circ f_n$ for each $n \in \mathbb{N}$ and $G = h \circ f$.

(a) Explain why G and G_n (for all $n \in \mathbb{N}$) are measurable. ***(2 marks)***

(b) Deduce that the sequence (G_n) converges pointwise almost everywhere to G. ***(5 marks)***

(v) (a) Let $a \in \mathbb{R}$. Explain why the mapping $x \to \dfrac{1}{a^2 + x^2}$ is integrable with respect to Lebesgue measure on $[0, \infty)$. ***(5 marks)***

(b) Deduce that the mapping $x \to \dfrac{e^{-bx}}{a^2 + x^2}$ is integrable with respect to Lebesgue measure on $[0, \infty)$, where $b > 0$. ***(3 marks)***

3 Throughout this question (S, Σ, m) is a measure space and $\mathbb{R}$ is equipped with its usual Borel σ-algebra.

(i) (a) Explain how to define $\int_S f dm$ in the case where $f : S \to \mathbb{R}$ is a non-negative *simple* function, i.e. $f = \sum_{i=1}^{n} c_i \mathbf{1}_{A_i}$ where $c_1, \ldots, c_n \in [0, \infty)$ and $A_1, \ldots A_n \in \Sigma$ for some $n \in \mathbb{N}$ with $\bigcup_{n=1}^{\infty} A_n = S$ and $A_i \cap A_j = \emptyset$ when $i \neq j$. ***(2 marks)***

(b) Explain how to extend the definition of $\int_S f dm$ to the case where $f : S \to \mathbb{R}$ is an arbitrary non-negative measurable function. What does it mean for such an f to be *integrable*? ***(3 marks)***

(c) Explain how to define $\int_S f dm$ in the case where $f : S \to \mathbb{R}$ is measurable but no longer necessarily non-negative. What does it mean for such an f to be *integrable*? ***(3 marks)***

(ii) State the *monotone convergence theorem* and use it to prove *Fatou's lemma*, i.e. if (f_n) is a sequence of non-negative functions from S to $\mathbb{R}$ then

$$\liminf_{n\to\infty} \int_S f_n dm \geq \int_S \liminf_{n\to\infty} f_n dm.$$

(8 marks)

(iii) (a) Deduce the *reverse Fatou lemma*, i.e. if (f_n) is a sequence of non-negative measurable functions for which $f_n \leq f$ for all $n \in \mathbb{N}$ where f is integrable then

$$\limsup_{n\to\infty} \int_S f_n dm \leq \int_S \limsup_{n\to\infty} f_n dm.$$

(5 marks)

[Hint. Apply Fatou's lemma to $f - f_n$.]

(b) Show that the reverse Fatou lemma fails to work in the case where S is the real number line equipped with Lebesgue measure and $f_n = \mathbf{1}_{(n,n+1]}$ for each $n \in \mathbb{N}$ and comment on why there is no contradiction here with the result just proved. ***(3 marks)***

 Question 3 continued on next page

3 (continued)

(iv) (a) Let (S, Σ, m) be a measure space and $f : [a, b] \times S \to \mathbb{R}$ be a measurable function for which

(I) The mapping $x \to f(t, x)$ is integrable for all $t \in [a, b]$,

(II) The mapping $t \to f(t, x)$ is continuous for all $x \in S$,

(III) There exists a non-negative integrable function $g : S \to \mathbb{R}$ so that $|f(t, x)| \leq g(x)$ for all $t \in [a, b], x \in S$.

Use the dominated convergence theorem to show that the mapping $t \to \int_S f(t, x) dm(x)$ is continuous on $[a, b]$. ***(6 marks)***

(b) If $f : \mathbb{R} \to \mathbb{R}$ is integrable, deduce that the mapping from $[0, 1]$ to $\mathbb{R}$ given by $t \to \int_{\mathbb{R}} \frac{f(x)}{1 + t} dx$ is continuous. ***(3 marks)***

4 Throughout this question all random variables are defined on a common probability space $(\Omega, \mathcal{F}, P)$.

(i) (a) Write the *expectation* $\mathbb{E}(X)$ of an integrable random variable X as a Lebesgue integral with respect to the measure P and then use expectation to define the variance $\mathrm{Var}(X)$. ***(2 marks)***

(b) If X and Y are integrable random variables so that $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$, explain briefly why integration theory enables us to conclude that $\mathbb{E}(X) \leq \mathbb{E}(Y)$. ***(2 marks)***

(ii) (a) Suppose that X and Y are random variables wherein both X^2 and Y^2 are integrable. Prove the *Cauchy-Schwarz inequality*:

$$|\mathbb{E}(XY)| \leq (\mathbb{E}(X^2)^{\frac{1}{2}})(\mathbb{E}(Y^2)^{\frac{1}{2}}).$$

(4 marks)

[Hint: Consider $g(t) = \mathbb{E}((X + tY)^2)$ as a quadratic function of $t \in \mathbb{R}$.]

(b) Deduce that if X^2 is integrable, then so is X and that

$$|\mathbb{E}(X)|^2 \leq \mathbb{E}(X^2).$$

(3 marks)

(iii) Deduce that for a random variable X having a finite mean μ, $\mathbb{E}(X^2) < \infty$ if and only if $\mathrm{Var}(X) < \infty$. Show further that $\mathbb{E}(|X - \mu|)^2 \leq \mathrm{Var}(X)$. ***(4 marks)***

(iv) Let X be a random variable for which $\mathbb{E}(e^{t|X|}) < \infty$ for all $t > 0$.

(a) Deduce that $\mathbb{E}(e^{tX}) < \infty$ for all $t \in \mathbb{R}$. ***(2 marks)***

(b) Show that $\mathbb{E}(|X^n|) < \infty$ for all $n \in \mathbb{N}$. ***(2 marks)***

(c) Prove that $\mathbb{E}(X) = \left.\frac{d}{dt}\mathbb{E}(e^{tX})\right|_{t=0}$, giving careful details of the use of appropriate convergence theorems. [Hint: Use the mean value theorem.] ***(6 marks)***

(d) The Poisson random variable N with parameter $c > 0$ takes values in $\mathbb{N} \cup \{0\}$ with probability law

$$P(N = k) = \frac{e^{-c}c^k}{k!},$$

for $k = 0, 1, 2, \ldots$ Deduce that $\mathbb{E}(e^{tN}) < \infty$ for all $t > 0$, find an explicit formula for $\mathbb{E}(e^{tN})$ and use the result of (iii) to find $\mathbb{E}(N)$. ***(8 marks)***

End of Question Paper