The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

Mathematical Methods **2 hours**

*Marks will be awarded for your best **FOUR** answers. The marks awarded to each question or section of question are shown in italics.*

1 A function $f(x)$ is defined for $-\infty < x < \infty$ by

$$f(x) = x\,e^{-x^2}.$$

(a) Show that the Fourier transform, $\hat{f}(k)$, of $f(x)$ is given by

$$\hat{f}(k) = \frac{1}{2}\,i\sqrt{\pi}\,k\,e^{-k^2/4}.$$ ***(8 marks)***

[You may assume that

$$\int_{-\infty}^{\infty} (x-a)\,e^{-(x-a)^2}\,dx = \int_{-\infty}^{\infty} x\,e^{-x^2}\,dx$$

and

$$\int_{-\infty}^{\infty} e^{-(x-a)^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi},$$

where a may be complex, but does not depend on x.]

(b) Show that applying the inverse Fourier transform to $\hat{f}(k)$ gives $f(x)$. ***(8 marks)***

(c) Verify that Parseval's theorem

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\hat{f}(k)\right|^2\,dk$$

holds for this $f(x)$. ***(9 marks)***

2 The function $x(t)$ satisfies the ordinary differential equation

$$\ddot{x} + \dot{x} - 6x = f(t)$$

for $t \geqslant 0$, for some function $f(t)$, with $x(0) = 1$ and $\dot{x}(0) = 2$.

(a) Taking the Laplace transform of the equation, find $\tilde{x}(s)$ in terms of $\tilde{f}(s)$, where $\tilde{x}(s)$ is defined by

$$\tilde{x}(s) = \int_0^\infty e^{-st}\, x(t)\, dt,$$

and $\tilde{f}(s)$ is defined similarly. ***(9 marks)***

Hence derive the solution

$$x(t) = e^{2t} + \frac{1}{5} \int_0^t f(u) \left\{ e^{2(t-u)} - e^{-3(t-u)} \right\} du.$$ ***(5 marks)***

> You may assume that the following hold:
>
> $$\mathcal{L}\left\{x^{(n)}(t)\right\} = s^n \tilde{x}(s) - s^{n-1}x(0) - s^{n-2}\dot{x}(0) - \cdots - x^{(n-1)}(0)$$
>
> $$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a} \quad \text{for} \quad \operatorname{Re} s > a,$$
>
> $$\mathcal{L}\left\{\int_0^t f(u)\, g(t-u)\, du\right\} = \tilde{f}(s)\tilde{g}(s),$$
>
> where $\mathcal{L}\{\cdot\}$ denotes the Laplace transform.

(b) Use the result of part (a) to find the solution $x(t)$ when $f(t) = e^{2t}$. ***(5 marks)***

Verify that this solution does satisfy the differential equation and the initial conditions. ***(6 marks)***

3 The function $y(x)$ satisfies the ordinary differential equation

$$x^2y'' + 2xy' - 2y = x^2e^{-2x}$$

in $0 < x < \infty$, with the boundary conditions that y is finite at $x = 0$ and as $x \to \infty$.

(a) By trying $y = x^n$, find the independent solutions of

$$x^2y'' + 2xy' - 2y = 0.$$

(3 marks)

(b) Given that Green's function, $G(x;\xi)$, for the boundary-value problem given at the beginning of the question is continuous at $x = \xi$, and that $\partial G/\partial x$ has a discontinuity of size $1/\xi^2$ at $x = \xi$, show that

$$G(x;\xi) = \begin{cases} -\dfrac{x}{3\xi^2} & 0 \leqslant x < \xi, \\ \\ -\dfrac{\xi}{3x^2} & \xi < x < \infty. \end{cases}$$

(10 marks)

(c) Using Green's function, show that the solution to the boundary-value problem given at the beginning of the question is

$$y(x) = \frac{1}{8}e^{-2x}\left(2 + \frac{2}{x} + \frac{1}{x^2}\right) - \frac{1}{8x^2}.$$

(12 marks)

4 Consider the equation

$$\epsilon x^3 + x^2 - 18 = 0, \tag{1}$$

where ϵ is a constant satisfying $0 < \epsilon \ll 1$.

(a) The solution can be written as

$$x = \frac{1}{\epsilon}\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots\right),$$

where $x_0, x_1, x_2, \ldots$ are $O(1)$ as $\epsilon \to 0$.

Substitute into equation (1) to derive the solutions for x, correct to order ϵ as $\epsilon \to 0$.

(19 marks)

(b) Given the rearrangement

$$x = \left(18 - \epsilon x^3\right)^{1/2}$$

of (1), use iteration to find the solution close to $3\sqrt{2}$, correct to order ϵ^2 as $\epsilon \to 0$.

(6 marks)

5 The complementary error function, $\operatorname{erfc}(x)$, is defined for $x > 0$ by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt.$$

By changing variables, show that

$$\operatorname{erfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi}} \int_0^\infty e^{-xv}\, e^{-v^2/4}\, dv.$$ ***(3 marks)***

Expand $e^{-v^2/4}$ in powers of v and change variables by $u^2 = xv$. Hence, defining

$$I_n = \int_0^\infty u^{2n+1}\, e^{-u^2}\, du,$$

show that

$$\sqrt{\pi}\, x\, e^{x^2} \operatorname{erfc}(x) \sim 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(4x^2)^k}\, I_{2k} \qquad \text{as } x \to \infty.$$ ***(7 marks)***

Show that $I_n = nI_{n-1}$ for $n > 0$, and hence find an asymptotic expansion for $\operatorname{erfc}(x)$ as $x \to \infty$. ***(10 marks)***

For the second term in the asymptotic expansion to make less than a 1% relative change to the first term, deduce that $x > \sqrt{50}$. ***(5 marks)***

End of Question Paper