Data provided: Formula sheet

MAS323

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

Differential Equations: Case Studies in Applied Mathematics **2 hours**

Attempt all questions.

1 A two-species predator-prey system with populations x and y respectively is modelled by the equations

$$\frac{dx}{dt} = Ax\left(1 - \frac{x}{K}\right) - Bxy(1 - e^{-Cx})$$

and

$$\frac{dy}{dt} = -Dy + Ey(1 - e^{-Cx}),$$

in which the six parameters (A, B, C, D, E, K) are strictly positive.

(i) What do each of the four terms on the righthand sides of these equations imply ecologically? ***(4 marks)***

(ii) Non-dimensionalise the system using the scalings

$$X = \frac{x}{K}, \quad Y = \frac{By}{A}, \quad T = At, \quad \alpha = \frac{E}{A}, \quad \delta = \frac{D}{A}, \quad \beta = CK,$$

to obtain the non-dimensional system given by

$$\frac{dX}{dT} = X(1 - X) - XY(1 - e^{-\beta X}),$$

and

$$\frac{dY}{dT} = -\delta Y + \alpha Y(1 - e^{-\beta X}).$$

(4 marks)

(iii) Determine the co-ordinates of the three critical points, noting any parameter restrictions. ***(7 marks)***

(iv) Determine the stability of each of the critical points. ***(10 marks)***

2 (i) (a) The Euler-Lagrange equation corresponding to a functional $F(x, y(x), y'(x))$ is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0.$$

Show that

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x}.$$

(4 marks)

(b) Hence, in the case that F is independent of x, show that

$$F - y'\frac{\partial F}{\partial y'} = \text{constant}.$$

(2 marks)

(ii) Find the function y that minimises the length

$$J[y] = \int_a^b \sqrt{1 + (y')^2}\ dx$$

subject to $y(a) = c$ and $y(b) = d$. ***(11 marks)***

(iii) When the curve $y = y(x)$ that joins two points, (x_0, y_0) and (x_1, y_1), is rotated once about the $y-$axis, the surface generated is

$$S[y] = 2\pi \int_{x_0}^{x_1} y(1 + y'^2)^{\frac{1}{2}}\ dx.$$

Show that an extremal for this problem is given by

$$y(x) = p\cosh\left(\frac{x}{p} + q\right),$$

where p and q are constants. ***(8 marks)***

3 (i) Consider the system of equations

$$\begin{aligned} \dot{x} &= y + xF(r), \\ \dot{y} &= -x + yF(r), \end{aligned} \qquad (*)$$

where $r^2 = x^2 + y^2$.

(a) By making the substitutions $x = r\cos\theta$ and $y = r\sin\theta$ show that this system has a periodic solution for each value of r_0 such that $F(r_0) = 0$. ***(10 marks)***

(b) You are given that this periodic solution is a stable limit cycle in the case $F'(r_0) < 0$ and that it is an unstable limit cycle in the case $F'(r_0) > 0$.
For the case $F(r) = -(r-3)(r^2 - 6r + 5)$ find all the limit cycles and determine their stability. ***(7 marks)***

(ii) A non-linear system is described by the equations

$$\frac{dx_1}{dt} = 9x_1 - x_1x_2^2 = F_1(x_1, x_2)$$

$$\frac{dx_2}{dt} = x_1^2 - x_2^2 = F_2(x_1, x_2).$$

(a) Find the five critical points of this system. ***(2 marks)***

(b) Linearise this system about the critical point for which both co-ordinates have positive (non-zero) values. ***(6 marks)***

4 The one-dimensional heat equation can be written as

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where the constant c^2 denotes the thermal diffusivity. This equation can be used to model the temperature distribution in a laterally insulated, homogeneous metal bar of length a.

(i) Use the technique of separating the variables, with $u(x,t) = F(x)G(x)$, to find the general solution of this equation that satisfies the boundary conditions

$$u(0,t) = 0\ ^\circ C, \quad u(a,t) = 0\ ^\circ C \quad \text{for all } t \geq 0.$$

(20 marks)

(ii) You are given that the initial temperature distribution along the bar is given by

$$u(x,0) = 100\left(\sin\frac{3\pi x}{80}\right)\ ^\circ C. \qquad (*)$$

If the length of the bar is 80 cm and $c^2 = 1.158\ \text{cm}^2\text{s}^{-1}$, calculate how long it will take for the maximum temperature in the bar to drop to 50 $^\circ C$. ***(5 marks)***

End of Question Paper

List of Basic Formulae and Theorems

Theorem 1: If a periodic solution of the system of equations

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y)$$

exists in a simply connected region, then $f_x + g_y = 0$ somewhere in that region.

Corollary: There are no periodic solutions in any simply connected region where $f_x + g_y \neq 0$ everywhere.

Theorem 2: The orbit $\mathcal{C}$ of a periodic solution must enclose at least one critical point.

Orthogonality conditions for trig functions

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad \text{when } \ m \neq n.$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0.$$

Extremals of functional

$$J[y] = \int_{x_0}^{x_1} f(y, y', x) \, dx$$

are the solutions to the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$