The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2013-2014**

Operations Research **2 Hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 Use the two-phase method to solve the following problem:

$$\text{Min } z = 2x_1 + 3x_2 - 5x_3$$
$$x_1 + x_2 + x_3 = 7$$
$$2x_1 - 5x_2 + x_3 \geq 10$$

where x_1, x_2 and x_3 are non-negative. Clearly state your optimal solution and optimal cost. (Hint: Not counting the pre-processing steps, you should be able to finish phase 1 with two simplex iterations, and phase 2 with one simplex iteration.)

(25 marks)

2 (i) Bill wants to celebrate his new job by watching every movie showing in cinemas in five nearby towns. If he travels to another town, he will stay there until he has watched all the movies on show there. The following table provides the information about the 9 movies on show and the towns:

Location	Movies	Round-trip miles	Cost per movie (£)
Town 1	1, 3, 4	0	7.95
Town 2	1, 6, 8	25	5.50
Town 3	2, 5, 6, 7	30	5.00
Town 4	1, 8, 9	28	7.00
Town 5	2, 4, 5, 7	40	4.95

The cost of driving is 75 pence per mile. Bill wishes to determine the towns he needs to visit to see all the movies while minimising his total cost. Defining binary variables x_i such that

$$x_i = \begin{cases} 1 & \text{if Bill visits town } i \\ 0 & \text{if Bill does NOT visit town } i \end{cases}$$

for $i = 1, 2, ..., 5$, formulate the mixed integer/linear programming problem. Do NOT solve the resulting problem. ***(18 marks)***

(ii) For each of the following sets of conditions, set up the appropriate mathematical expressions suitable for integer linear programming:

(a) If $g^T x > c$, then $f^T x \geq d$ where g and f are two constant vectors, and c and d are constants.

(b) Either ($x_1+2x_2 \leq 6$ is imposed) or (both $2x_1+x_2 \leq 8$ and $x_1-x_2 \geq 3$ are imposed).

(7 marks)

3 A company produces units of A, B and C, using labour and raw materials M_1 and M_2. The requirements for per unit of each product are given as follows:

	M_1(kg)	M_2(kg)	Labour(hour)	Profit(£)
A	3	8	2	69
B	6	12.5	3	111
C	2	3.5	1	42

The company has available in the next month 1 ton of M_1, 2.5 tons of M_2, and 550 hours of labour. To determine the optimal production schedule, the manager defines x_1, x_2 and x_3 as the number of A's, B's, and C's to be produced in the next month, and formulates the following model for the optimal profit:

$$\text{Maximise} \quad z = 69x_1 + 111x_2 + 42x_3$$

subject to

$$3x_1 + 6x_2 + 2x_3 \leq 1000$$
$$8x_1 + 12.5x_2 + 3.5x_3 \leq 2500$$
$$2x_1 + 3x_2 + x_3 \leq 550 \ .$$

Using slack variables x_4, x_5, and x_6, the optimal tableau is found as follows:

Basis	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	15	0	15	0	12	21600
x_3	0	3	1	2	0	-3	350
x_5	0	2	0	1	1	-5.5	475
x_1	1	0	0	-1	0	2	100

(i) What is the optimal production schedule and the corresponding profit? ***(2 marks)***

(ii) What are the optimal values of the dual variables? What do they tell us about the constraints? ***(3 marks)***

(iii) Suppose the amount of raw material M_1 used in B can be reduced from the current 6kg/unit requirement, causing no other changes in the data in the problem. By how much does it need to be reduced to have B introduced into the optimal production schedule? ***(6 marks)***

(iv) The manager is contemplating manufacturing another product D. A unit of D requires 4kg of M_1, 5kg of M_2, and 1.5 hours of labour. What is the minimum profit for D so that it is profitable to be included in the optimal production schedule? ***(5 marks)***

(v) Denote the amount of available raw material M_1 by b_1. Find the range of b_1 for which the optimal production schedule remains unchanged. ***(7 marks)***

(vi) Suppose additional M_1 can be purchased at a cost of £7.75/kg. Should more be purchased and why? ***(2 marks)***

4 The payoff matrix for a two-person zero-sum game is given as follows:

$$A = \begin{bmatrix} 1 & -4 \\ -3 & 4 \\ -2 & 3 \\ 0 & -5 \end{bmatrix},$$

where the rows represent the pure strategies for player A and the columns represent those for player B.

(i) Show that the game has no pure strategy equilibrium solution.
(2 marks)

(ii) Set up the two linear programming problems from which you can solve the optimal mixed strategy solutions for both players. Do NOT solve the resulting problem.
(4 marks)

(iii) Find the dominated row strategy and hence reduce the payoff matrix to a 3×2 matrix.
(2 marks)

(iv) Using the reduced payoff matrix, find the optimal strategies for the players and the value of the game (you may use graphs to assist your calculation).
(17 marks)

5 (i) Given the following linear programming problem:

$$\begin{aligned} \text{Maximise} \quad & z = 5x_1 + 12x_2 + 4x_3 \\ \text{subject to} \quad & \\ & x_1 + 2x_2 + x_3 \leq 10 \\ & 2x_1 - x_2 + 3x_3 = 8 \\ & x_1, x_2, x_3 \geq 0, \end{aligned}$$

write down the canonical form of the problem, and hence find the dual problem. ***(5 marks)***

(ii) Write down the complementary slackness conditions for the primal and dual linear programming problems:

$$\begin{aligned} &\text{Max } z(x) = c^T x, \quad Ax \leq b, \quad x \geq 0, \\ &\text{Min } w(y) = b^T y, \quad A^T y \geq c, \quad y \geq 0. \end{aligned}$$

Give an interpretation of the conditions in terms of shadow costs and reduced costs. ***(6 marks)***

(iii) Show that $w(y) \geq z(x)$ for any feasible solutions x and y. ***(6 marks)***

(iv) For the following linear programming problem

$$\begin{aligned} \text{Maximise} \quad & z = x_2 + 2x_3 \\ \text{subject to} \quad & \\ & x_1 - 2x_2 + 2x_3 \leq -8 \\ & -x_1 + x_2 + x_3 \leq 5 \\ & 2x_1 - x_2 + 4x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0, \end{aligned}$$

introducing slack variables x_4, x_5, $x_6 \geq 0$, use the dual simplex algorithm to show that $x_2 = 4$, $x_5 = 1$, $x_6 = 14$, $x_1 = x_3 = x_4 = 0$ is a feasible basic solution (Hint: you need only perform one dual simplex step). ***(8 marks)***

End of Question Paper