The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2013-2014**

Applied Differential Equations **2 hours**

Attempt all FOUR questions.

1 (i) Show that the following linear multi-step method

$$y_{n+1} = y_{n-1} + \frac{h}{12}(5f_{n+1} + 31f_n - 17f_{n-1} + 5f_{n-2}),$$

is consistent and stable, where h is the step-size, $f_n = f(x_n, y_n)$ and $y'(x) = f(x, y)$.

(12 marks)

(ii) The following table contains grid-point values of two solutions $Y_1(x)$ and $Y_2(x)$ of a linear differential equation $d^2y/dx^2 = f(x, y, y')$ obtained using the fourth-order Runge-Kutta method.

x	1.5	2.0	2.5	3.0
$Y_1(x)$	1.07444	1.36280	2.02865	3.41281
$Y_2(x)$	1.36222	2.05472	3.35550	5.85592

$Y_1(x)$ was determined using the initial conditions $y(1) = 1$, $y'(1) = 0$, and $Y_2(x)$ was obtained using $y(1) = 1$, $y'(1) = 0.5$. Form a linear combination of these two solutions which is the numerical solution to the equation $d^2y/dx^2 = f(x, y, y')$ with boundary conditions $y(1) = 1$, $y(3) = 2$. Calculate the values of this solution at all the x-values given in the table.

(7 marks)

(iii) The Taylor series expansion for $y(x_{n+1})$, where $x_{n+1} = x_n + h$, is given as

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + \frac{h^3}{6}y'''(x_n) + O(h^4).$$

Write down the expansion for $y(x_{n-1})$, where $x_{n-1} = x_n - h$, and then derive the relation

$$y''(x_n) = \frac{y(x_{n+1}) - 2y(x_n) + y(x_{n-1})}{h^2} + O(h^2).$$

(6 marks)

2 A single step method for the equation $y' = f(x, y)$ is defined by the following formulas:

$$k_1 = hf_n, \quad k_2 = hf\left(x_n + \frac{5}{6}h, y_n + \frac{5}{6}k_1\right),$$
$$y_{n+1} = y_n + \frac{2}{5}k_1 + \frac{3}{5}k_2\ ,$$

where $f_n = f(x_n, y_n)$.

(i) Write down its local discretisation error $\tau(x_n, h)$. By finding the limit of $\tau(x_n, h)$ when $h \to 0$, show that the method is consistent. ***(8 marks)***

(ii) We want to use the method to solve the equation

$$y'(x) = -0.2y(x), \quad y(0) = 1. \tag{1}$$

Find the largest possible step-size h in order that the method is absolutely stable. ***(10 marks)***

(iii) Two approximate solutions at $x = 3$ have been found using the above method with two step-sizes. The results are:

h	Approximate solutions for $y(3)$
0.3	0.5490
0.6	0.5497

Without using the analytic solution of Equation (1), use the data in the table to estimate the step-size required to ensure that the absolute value of the global discretization error in $y(3)$ is smaller than 10^{-5}. You may assume that the above method is of order 2. Work throughout with four decimal places. ***(5 marks)***

(iv) Estimate the magnitude of the global discretisation error of the solution at $x = 3$ if the step-size is $h = 0.9$. ***(2 marks)***

3 (i) Consider Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for $0 \le x \le 1$ and $0 \le y \le 1$, with boundary conditions

$$u(0, y) = 0, \quad -u(x, 0) + 2\left.\frac{\partial u(x, y)}{\partial y}\right|_{y=0} = 0, \quad u(x, 1) = 0,$$

and $u(1, y)$ unspecified.

(a) Letting $u(x, y) = X(x)Y(y)$ be a separable solution for the above equation, show that

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \alpha.$$

Explain why α must be a constant. ***(3 marks)***

(b) Given that there is only a trivial separable solution when $\alpha \le 0$, show that the non-trivial solution has the following expression

$$u(x, y) = C \sinh(sx)[2s\cos(sy) + \sin(sy)],$$

where C is a constant and s is a root of the following algebraic equation

$$2s\cos(s) + \sin(s) = 0.$$

(16 marks)

(ii) Show that, with variable substitutions $\eta = x + ct$, $\nu = x - ct$, the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + g$$

becomes

$$\frac{\partial^2 u}{\partial \eta \partial \nu} = -\frac{g}{4c^2},$$

where g is a constant. ***(6 marks)***

4 Consider the following equation for $u(x,t)$

$$\frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2},$$

for $0 \leq x \leq 1$ and $t \geq 0$, subject to inhomogeneous boundary conditions

$$u(0,t) = 1, \quad u(1,t) = e^2,$$

and initial condition

$$u(x,0) = e^x \sin(2\pi x) + e^{2x}.$$

(i) Letting $u(x,t) = v(x,t) + e^{2x}$, find the equation for $v(x,t)$, and its boundary and initial conditions. ***(6 marks)***

(ii) Use separation of variables to find the solution for $u(x,t)$. You may use the following result: If $a^2 - 4b \geq 0$, and the boundary conditions $X(0) = X(1) = 0$ are imposed, then the equation

$$X''(x) + aX'(x) + bX(x) = 0$$

has only the trivial solution. ***(19 marks)***

End of Question Paper