Data provided: Formula Sheet

MAS420

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2013–14

MAS420 Signal Processing — 2 hours

Answer four *questions. If you answer more than four questions, only your best four will be counted.*

1 (i) For the Hilbert space $L^2[0,3]$:

(a) Find $||f||$, where $f(t)$ is the signal t, and find $||g||$ where $g(t)$ is the signal $e^{i\omega t}$. ***(2 marks)***

(b) Find the inner product $(t, e^{i\omega t})$. ***(3 marks)***

(ii) State (without proof) the Cauchy-Schwarz inequality. State the condition for equality. ***(2 marks)***

Use the Cauchy-Schwarz inequality on an appropriate space, which you should define, to show that if $f(t)$ is a real signal of energy 16, then

$$\left|\int_0^1 tf(t)dt\right|^2 \leq \frac{16}{3}.$$ ***(5 marks)***

Find a signal, $f(t)$, that gives equality. ***(2 marks)***

(iii) The time-bandwidth theorem states that, for an Ω-bandlimited signal for which the equivalent rectangle resolution, τ, is defined, then $\tau\Omega \geq \pi$.

(a) Show that the signal

$$f(t) = \text{sinc}(3t)\cos t$$

satisfies the conditions of the time-bandwidth theorem. ***(2 marks)***

(b) Find the equivalent rectangle resolution for $f(t)$ and verify that $\tau\Omega > \pi$. ***(9 marks)***

2 (i) (a) Sketch the function

$$f(t) = \begin{cases} \cos \pi t & : \quad |t| \leq 2 \\ 0 & : \quad |t| > 2. \end{cases}$$

(2 marks)

(b) Use direct integration to show that the Fourier transform, $F(\omega)$, of $f(t)$, assuming that $|\omega| \neq \pi$, is given by

$$F(\omega) = -\frac{2\omega \sin 2\omega}{\pi^2 - \omega^2}.$$

(9 marks)

(c) Hence find the Fourier transform of $f(t)\cos \omega_0 t$. ***(2 marks)***

(ii) Find the energy in the signal $g(t) = e^{-t}U(t)$ and find the bandwidth, W, in rad s^{-1}, such that 95% of the energy is contained in frequencies below W. ***(12 marks)***

3 (i) Assuming the Fourier transform pair $\overline{\delta}_T(t) \leftrightarrow \sigma\overline{\delta}_\sigma(\omega)$, where $\overline{\delta}_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$, $\overline{\delta}_\sigma(\omega)$ is defined similarly and $\sigma = 2\pi/T$, prove that

$$f_s(t) \equiv f(t)\overline{\delta}_T(t) \leftrightarrow \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - n\sigma),$$

where $F(\omega)$ is the Fourier transform of $f(t)$. ***(5 marks)***

(ii) Using the previous result, show that if $f(t)$ is Ω-bandlimited and $T < \pi/\Omega$, then $f(t)$ can be recovered exactly from the sampled signal $f_s(t)$ by the sinc interpolation formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT)\,\text{sinc}\left\{\frac{\sigma}{2}(t - nT)\right\}.$$

Clear diagrams are likely to help your answer. ***(6 marks)***

(iii) Find the Nyquist frequency, in Hz, of the signal

$$f(t) = \text{sinc}(\pi t).$$

(4 marks)

(iv) This signal is sampled at 3/4 of the Nyquist frequency and the samples are used to form a signal $g(t)$ by sinc interpolation. Making use of clear diagrams, find $G(\omega)$ and hence $g(t)$. ***(7 marks)***

(v) Verify that $f(t) \neq g(t)$ by considering their respective values at $t = 1$. ***(3 marks)***

4 (i) Sketch the two-sided exponential pulse given by

$$f(t) = \begin{cases} e^{at} & : \quad t \leq 0 \\ e^{-at} & : \quad t > 0, \end{cases}$$

where $a > 0$. ***(2 marks)***

Using direct integration, show that the Fourier transform of $f(t)$ is $\dfrac{2a}{a^2 + \omega^2}$. ***(8 marks)***

(ii) With the aid of clear diagrams, and without using Fourier transforms, find the convolution of the rectangular pulse, $p_1(t)$, with the function $f(t)$. ***(11 marks)***

(iii) Deduce the Fourier transform of the function $g(t) = \dfrac{4a\,\mathrm{sinc}(t)}{a^2 + t^2}$. ***(4 marks)***

5 (i) (a) Define the terms 'linear operator system' and 'shift-invariant operator'. ***(2 marks)***

(b) The function $g(t)$ is the response to an input, $e^{\alpha t}$, where $\alpha \in \mathcal{C}$. Prove that if S is a linear shift-invariant (LSI) operator, then

$$S(e^{\alpha t}) = K(\alpha)e^{\alpha t},$$

where K is dependent on α, but is independent of t. ***(5 marks)***

(c) Use a suitable choice of α in the above to define the system transfer function, $H(\omega)$.
Define the impulse response function, $h(t)$, of a LSI system and state the relationship between this and the system transfer function (STF). ***(3 marks)***

(d) Find the STFs of the systems whose impulse responses are given by:

$h(t) = \frac{1}{2}\,\mathrm{sinc}(5t),$

$h(t) = e^{-7t}U(t),$

$h(t) = \sin 2t.$ ***(4 marks)***

(e) For the case when $h(t) = \frac{1}{2}\,\mathrm{sinc}(5t)$, find the response if the input is

$$\sin t + \cos 6t.$$

(4 marks)

(ii) The function $f(t)$ is defined by

$$f(t) = \cos\omega_0 t + 2\cos 3\omega_0 t + 3\cos 4\omega_0 t.$$

(a) Write down the Fourier transform, $F(\omega)$, of $f(t)$. ***(2 marks)***

(b) Sketch $F(\omega)$. ***(2 marks)***

(c) A low pass filter, $H(\omega) = q_{2\omega_0}(\omega)$, is applied. Sketch the resulting filtered signal. ***(3 marks)***

End of Question Paper

Formula sheet

Function Definitions:

Rectangular pulse:

$$p_a(t) = \begin{cases} 1 & |t| \leq a \\ 0 & |t| > a \end{cases}$$

Triangular pulse:

$$q_a(t) = \begin{cases} 1 - |t|/a & |t| \leq a \\ 0 & |t| > a \end{cases}$$

Step function:

$$U(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Fourier Transform Pairs:

$$\begin{aligned}
p_a(t) &\longleftrightarrow 2a\,\mathrm{sinc}(a\omega) \\
q_a(t) &\longleftrightarrow a\,\mathrm{sinc}^2(a\omega/2) \\
\mathrm{sinc}(at) &\longleftrightarrow \frac{\pi}{a} p_a(\omega) \\
\mathrm{sinc}^2(at) &\longleftrightarrow \frac{\pi}{a} q_{2a}(\omega) \\
\mathrm{e}^{-at}U(t) &\longleftrightarrow \frac{1}{a + i\omega} \\
\delta(t) &\longleftrightarrow 1 \\
\delta(t - t_0) &\longleftrightarrow \mathrm{e}^{-i\omega t_0} \\
1 &\longleftrightarrow 2\pi\delta(\omega) \\
\mathrm{e}^{i\omega_0 t} &\longleftrightarrow 2\pi\delta(\omega - \omega_0) \\
\mathrm{e}^{-t^2/2\sigma^2} &\longleftrightarrow \sigma\sqrt{2\pi}\mathrm{e}^{-\sigma^2\omega^2/2}
\end{aligned}$$

Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,\mathrm{d}t$$

Inverse Fourier transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{i\omega t}\,\mathrm{d}\omega$$

Duality theorem: If $f(t) \longleftrightarrow F(\omega)$ then $F(t) \longleftrightarrow 2\pi f(-\omega)$

Scaling: If $f(t) \longleftrightarrow F(\omega)$ then $f(at) \longleftrightarrow \frac{1}{|a|}F(\omega/a)$.

Translation: If $f(t) \longleftrightarrow F(\omega)$ then $f(t - t_0) \longleftrightarrow \mathrm{e}^{-i\omega t_0}F(\omega)$.

Frequency Shift: If $f(t) \longleftrightarrow F(\omega)$ then $\mathrm{e}^{i\omega_0 t}f(t) \longleftrightarrow F(\omega - \omega_0)$

Fourier Series: If $f_T(t)$ is periodic with period T then, with $\sigma = 2\pi/T$, the complex Fourier series is

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{in\sigma t}$$

where

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f_T(t) \mathrm{e}^{-in\sigma t} \, \mathrm{d}t$$

Likewise, the real Fourier series is

$$f_T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\sigma t + b_n \sin n\sigma t)$$

where

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f_T(t) \cos n\sigma t \, \mathrm{d}t$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f_T(t) \sin n\sigma t \, \mathrm{d}t$$

Parseval's Theorem: If V is a Hilbert space, $\{\phi_n\}$ is an orthonormal basis for V and $f = \sum_n c_n \phi_n$, then

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Plancherel's Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$\int_{-\infty}^{\infty} f(t) g^*(t) \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) G^*(\omega) \, \mathrm{d}\omega$$

Energy Theorem: If $f(t) \longleftrightarrow F(\omega)$ then

$$\int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, \mathrm{d}\omega$$

Convolution Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f * g(t) = \int_{-\infty}^{\infty} f(s) g(t-s) \, \mathrm{d}s \longleftrightarrow F(\omega) G(\omega)$$

Product Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t) g(t) \longleftrightarrow \frac{1}{2\pi} F(\omega) * G(\omega).$$