The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2013-2014**

Complex Analysis **2 hours 30 minutes**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Express both of the following in the form $x + iy$:

$$\frac{13-i}{1-2i}\,; \qquad (1-i)^{11}\,.$$ ***(4 marks)***

(ii) Express

$$\frac{(1-i)^{13}}{(\sqrt{3}-i)^{11}}$$

in the form $re^{i\theta}$ with $r > 0$ and $-\pi < \theta \leq \pi$. ***(4 marks)***

(iii) State, without proof, the triangle inequalities for $|\,z + w|$ and $|\,z - w|$.

Show that, if $|\,z| \leq 1$, then

$$\frac{1}{5} \leq \left|\frac{3z-4}{2z+3}\right| \leq 7\,.$$ ***(4 marks)***

(iv) Write down the definitions of $\cosh z$ and $\sinh z$.

Find all the solutions of the following equation:

$$2\cosh z + \sinh z = i.$$ ***(5 marks)***

(v) The path γ is the arc of the circle $|\,z + 1| = 1$ from 0 to -2 given by $z = -1 + e^{it}$ $(0 \leq t \leq \pi)$. Evaluate

$$\int_\gamma \bar{z}\,dz\,, \qquad \int_\gamma z^3 \cos(z^4)\,dz\,.$$ ***(4 marks)***

(vi) Find all the sixth roots of -1. Hence express $x^6 + 1$ as the product of three real quadratic factors. ***(4 marks)***

2 (i) State, without proof, the Cauchy-Riemann equations for a differentiable function. ***(1 mark)***

(a) Let $g(z) = 4z - 3\overline{z}$ for all $z \in \mathbb{C}$. Prove that g is nowhere differentiable. ***(3 marks)***

(b) The function h is analytic in the complex plane and

$$\mathrm{Im}(h(z)) \;+\; \mathrm{Re}(h(z)) \;=\; 2 \quad \text{for all} \quad z \in \mathbb{C}\,.$$

Show that h is constant. ***(5 marks)***

(ii) In each of the following cases, determine whether there is a function k analytic on $\mathbb{C}$ with $\mathrm{Re}\,(k(x+iy)) = u(x,y)$, giving reasons for your answers:

(a) $u(x,y) \;=\; \cosh x \cosh y\,,$

(b) $u(x,y) \;=\; x^3 - 3xy^2 - 2y + 1\,.$

When k exists, find an explicit expression for $k(z)$ in terms of z and show that you have found all functions the satisfying the conditions. ***(8 marks)***

(iii) Let the path α from 1 to -3, consist of the straight line segment from 1 to $1+3i$, followed by the straight line segment from $1+3i$ to $-3+3i$, followed by the straight line segment from $-3+3i$ to -3 . Sketch α. Use the ML estimate to show that

$$\left| \int_\alpha \frac{e^z \sin z}{z^2}\, dz \right| \;\leq\; 10\, e \cosh 3.$$

(8 marks)

3 State, without proof, Cauchy's Theorem and Cauchy's Integral Formulae for a function and for its derivatives. Your statement should include conditions under which the results are valid. ***(7 marks)***

Let γ be the square contour with **vertices** $2, 2i, -2, -2i$ described in the anti-clockwise direction. Without using the Residue Theorem, evaluate

(i) $\displaystyle\int_\gamma \frac{\sin(\pi z)}{3z-1}\,dz\,,$ (ii) $\displaystyle\int_\gamma \frac{e^z+1}{z^2+9}\,dz\,,$

(iii) $\displaystyle\int_\gamma \frac{e^z}{z^2\,(z+3)}\,dz\,,$ (iv) $\displaystyle\int_\gamma \frac{e^z}{z(z+1)}\,dz\,.$

(14 marks)

Let the contour α be the circle $|z-1| = 2$ described in the positive direction. Evaluate

$$\int_\alpha (z^2+\bar{z})\,dz\,.$$

(4 marks)

4 (i) Let f have a pole of order k at α. Prove that the residue of f at the point α is given by

$$\mathrm{Res}\,\{f\,;\alpha\} = \frac{1}{(k-1)!}\lim_{z\to\alpha}\frac{d^{k-1}}{dz^{k-1}}[(z-\alpha)^k f(z)].$$ ***(5 marks)***

(ii) For each of the following functions, find **all the singularities** in $\mathbb{C}$. Classify these singularities giving reasons for your answers and evaluate the residue at each of them:

(a) $\dfrac{\cos(\pi z)}{e^z\,(z-1)^2}$, ***(4 marks)***

(b) $z\exp\left(\dfrac{1}{z-1}\right)$, ***(4 marks)***

(c) $\dfrac{e^{\pi z}}{e^{\pi z}+1}$, ***(5 marks)***

(d) $\dfrac{1+\cos(\pi z)}{(z-1)^2}$, ***(3 marks)***

(e) $\dfrac{1+\cos(\pi z)}{(z-1)^5}$. ***(4 marks)***

5 (i) State, without proof, Cauchy's Residue Theorem. Your statement should include conditions under which the result is valid. ***(4 marks)***

Let γ be the triangular contour with **vertices** 2, $2i$, $-2i$ described in the anti-clockwise direction. Evaluate

$$\int_\gamma \frac{\sin \pi z}{(2z+1)\cos \pi z}\, dz \,, \qquad \int_\gamma (z+1)\, \cos\left(\frac{1}{z-1}\right) dz \,.$$ ***(11 marks)***

(ii) Prove that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)(x^2+4)}\, dx \quad = \quad \frac{\pi(e-1)}{3e^2} \,.$$ ***(10 marks)***

End of Question Paper