



Answer **four** questions. If you answer more than four questions, only your best four will be counted.

Throughout this paper, unless otherwise stated, all normed vector spaces are either over the field of real numbers, \mathbb{R} , or the field of complex numbers, \mathbb{C}

1 (i) Let V be a normed vector space over the field \mathbb{R} . Say what is meant by a bounded linear map $f: V \rightarrow \mathbb{R}$, and define the norm of such a map. Prove that this norm satisfies the axioms required to turn the dual space $V^* = \text{Hom}(V, \mathbb{R})$ into a normed vector space. (9 marks)

(ii) Prove that the above dual space V^* is complete. (7 marks)

(iii) State the Hahn-Banach theorem. (3 marks)

(iv) Prove that the linear map $\tau: V \rightarrow (V^*)^*$ defined by the formula

$$\tau(v)(f) = f(v) \quad f \in V^*, v \in V$$

is an isometry. (6 marks)

2 (i) Let V and W be normed vector spaces. Say what is meant by the statement that a linear map $T: V \rightarrow W$ is *bounded*, and prove that if T is a bounded linear map, then the kernel, $\ker T$, is closed. (4 marks)

(ii) Let $C^\infty[0, 1]$ be the vector space of all real-valued infinitely-differentiable functions $f: [0, 1] \rightarrow \mathbb{R}$, equipped with the norm

$$\|f\| = \sup\{|f(t)| \mid t \in [0, 1]\}.$$

Let $C_0^\infty[0, 1]$ be the subspace of all infinitely differentiable functions $f: [0, 1] \rightarrow \mathbb{R}$ such that $f(0) = 0$.

(a) Prove that we have a bounded linear map $T: C^\infty[0, 1] \rightarrow C_0^\infty[0, 1]$ defined by the formula

$$(Tf)(x) = \int_0^x f(t) dt.$$

(4 marks)

(b) Prove that the above map T is bijective. (3 marks)

(c) Does the map T have a *continuous* inverse? Justify your answer. (4 marks)

(iii) (a) Say what is meant by an open map, and state the open mapping theorem. (3 marks)

(b) Prove that $C_0^\infty[0, 1]$ is a closed subspace of the space $C^\infty[0, 1]$? (4 marks)

(c) Is the normed vector space $C^\infty[0, 1]$ complete? Justify your answer. (3 marks)

3 (i) (a) State Zorn's lemma, including definition of the terms *maximal* and *upper bound*. (4 marks)

(b) Define what is meant by an orthonormal basis for a Hilbert space H . (3 marks)

(c) Prove that every Hilbert space has an orthonormal basis. (7 marks)

(ii) (a) For a function $f: \mathbb{R} \rightarrow \mathbb{C}$, define $Jf: (-\pi, \pi) \rightarrow \mathbb{C}$ by

$$Jf(x) = \frac{1}{\sqrt{2}}(\tan(x/2) + i)f(\tan(x/2)).$$

Show that the above formula describes a unitary operator $J: L^2(\mathbb{R}) \rightarrow L^2(-\pi, \pi)$. (7 marks)

(b) Use the fact that the functions

$$e_k(x) = \frac{1}{\sqrt{2\pi}} \exp(ikx) \quad k \in \mathbb{Z}$$

form an orthonormal basis of $L^2(-\pi, \pi)$ to prove that the functions

$$f_n(x) = \frac{(i-x)^n}{\sqrt{\pi}(i+x)^{n+1}}$$

form an orthonormal basis of $L^2(\mathbb{R})$. (4 marks)

4 (i) Let H be a Hilbert space. State what is meant by the statements that a map $T: H \rightarrow H$ is a *self-adjoint operator*, and that a map $U: H \rightarrow H$ is a *unitary operator*. (4 marks)

(ii) Let $A: H \rightarrow H$ be a bounded linear operator. Show that we have a well-defined operator

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \quad (A^0 = I).$$

(6 marks)

(iii) Let $n \in \mathbb{N}^{\geq 0}$. Show that $(e^A)^n = e^{nA}$. (6 marks)

(iv) Show that the operator e^A is invertible (even if A is not invertible), and $(e^A)^{-1} = e^{-A}$. (5 marks)

(v) Let $A: H \rightarrow H$ be self-adjoint. Show that e^{iA} is unitary. (4 marks)

5 (i) (a) Let A be a unital Banach algebra, and let $x \in A$ be an element such that $\|x\| < 1$. Prove that the element $1 - x$ is invertible. **(8 marks)**

(b) Define the *spectrum* of an element $x \in A$. Prove this it is a bounded subset of \mathbb{C} . **(5 marks)**

(c) State the spectral mapping theorem for polynomials. **(2 marks)**

(ii) Let A be the Banach algebra of all bounded linear maps $\ell^1 \rightarrow \ell^1$. Determine the spectrum of the following operators from ℓ^1 to ℓ^1 .

(a) $S(a_1, a_2, a_3, \dots) = (a_2, a_3, \dots)$.

(b) $T(a_1, a_2, a_3, \dots) = (a_3 - 2a_1, a_4 - 2a_2, a_5 - 2a_3, \dots)$.

[Hint: For part (a), find the eigenvalues of S . You may use without proof the fact that the spectrum of S is closed.]

(10 marks)

6 (i) Define what is meant by the statement that a linear map between normed vector spaces is a *compact operator*. **(2 marks)**

(ii) Let $K: V \rightarrow W$ be a compact operator between normed vector spaces V and W . Let (x_n) be a bounded sequence in V . Prove that (Kx_n) has a convergent subsequence. **(4 marks)**

(iii) Prove that any bounded linear map with finite-dimensional image is compact. **(4 marks)**

(iv) Prove that the operator $S: \ell^2 \rightarrow \ell^2$ defined by the formula

$$S(a_1, a_2, a_3, \dots) = (a_1, \frac{a_2}{2}, \frac{a_3}{3}, \dots)$$

is compact.

[You may use without proof here the fact that a norm-limit of a sequence of compact operators is again compact] **(5 marks)**

(v) What is the definition of a Fredholm operator? **(2 marks)**

(vi) Define $T: \ell^2 \rightarrow \ell^2$ by the formula

$$T(a_1, a_2, a_3, \dots) = (a_1 + a_3, \frac{a_2}{2} + a_4, \frac{a_3}{3} + a_5, \dots)$$

Show that T is Fredholm, and calculate $Index(T)$. You may use any standard results from the theory of Fredholm operators to do this. **(8 marks)**

End of Question Paper