The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012–2013**

Introductory Mathematical Finance and Time Series **3 hours**

Marks will be awarded for your best ***five*** *answers.*

Questions 1-3 are CLOSE BOOK EXAMINATION.
Questions 4-6 are RESTRICTED OPEN BOOK EXAMINATION
For Questions 4-6 candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator that conforms to University regulations.
There are 100 marks available on the paper.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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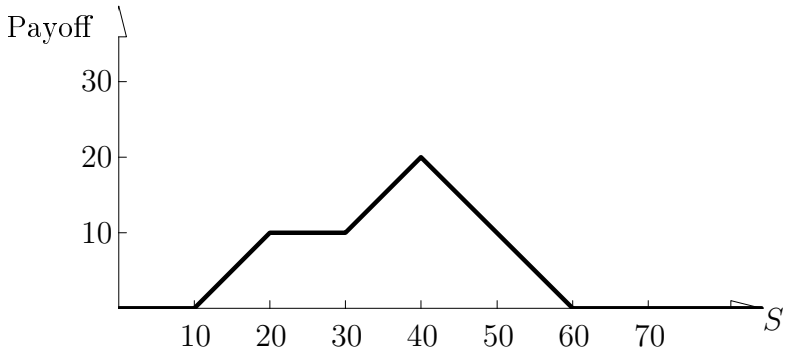
1 (i) Consider a twelve-month forward contract on one share of *ABC Plc*. These shares are currently traded for £10 per share, and spot interest rates for all maturities are 2%. Within the next twelve months, *ABC Plc* will pay a single dividend of 50p per share in 6 months.

(a) Find the present value of the dividend paid in 6 months and the twelve-month forward price of one share of *ABC Plc*. ***(2 marks)***

(b) You are given the opportunity to take a short position in this forward contract at a forward price of £9.75. Describe in detail an arbitrage opportunity available to you. ***(9 marks)***

(ii) (a) Describe a portfolio consisting entirely of European call options on the same stock, with same expiration time $T > 0$, but with different strike prices, and whose payoff at time T as a function of S, the spot price of the stock at time T, is described by the graph below. ***(5 marks)***



(b) Let $c_{10}, c_{20}, c_{30}, c_{40}$ and c_{60} be the prices of the above call options with strike prices $10, 20, 30, 40$ and 60, respectively, and let p_{60} be the price of a European put option on the same stock, with expiration at time T and with strike price 60. By comparing the payoff of the portfolio in (a) and the payoff of the put option above, describe an inequality involving $c_{10}, c_{20}, c_{30}, c_{40}, c_{60}$ and p_{60}. ***(4 marks)***

2 (i) The price of a stock which pays no dividends is currently £8. Over each of the next two 1-year periods the stock price will either increase by 50% or decrease by 50%. Suppose that all interest rates are constant and equal to 3%.

(a) Consider a one-year European call option on this stock with strike price £10. Describe a portfolio consisting of an amount of stock and some number of these call options whose value in one year is known with certainty. ***(4 marks)***

(b) Use a binomial tree to find the price of a two-year American put option on this stock with strike price £9. ***(6 marks)***

(c) Describe all circumstances when a rational investor should exercise the option described in (b) before its expiration. ***(2 marks)***

(ii) (a) State the mathematical definition of Brownian motion. ***(5 marks)***

(b) State Ito's Lemma. ***(3 marks)***

3 (i) Consider a world where there are only two risky investments: *Greed Plc* and *Safety First Inc.* stocks.

	Number of shares	Price per share	Expected return	Standard deviation of return
Greed Plc	1,000	£20	20%	50%
Safety First Inc.	2,000	£10	5%	10%

The correlation between the returns of these two stocks is 0.25.

(a) What is the market portfolio? ***(2 marks)***

(b) What are the expected return and standard deviation of returns of the market portfolio? ***(4 marks)***

(c) Find the beta coefficient of *Greed Plc.* ***(4 marks)***

(d) What should be the risk-free return in this world, if one existed? ***(4 marks)***

(e) Assume that risk-free deposits are available. Of all portfolios with expected returns of 8% consisting of a combination of risk-free investments and the two risky investments above, which one has the lowest standard deviation of returns. ***(6 marks)***

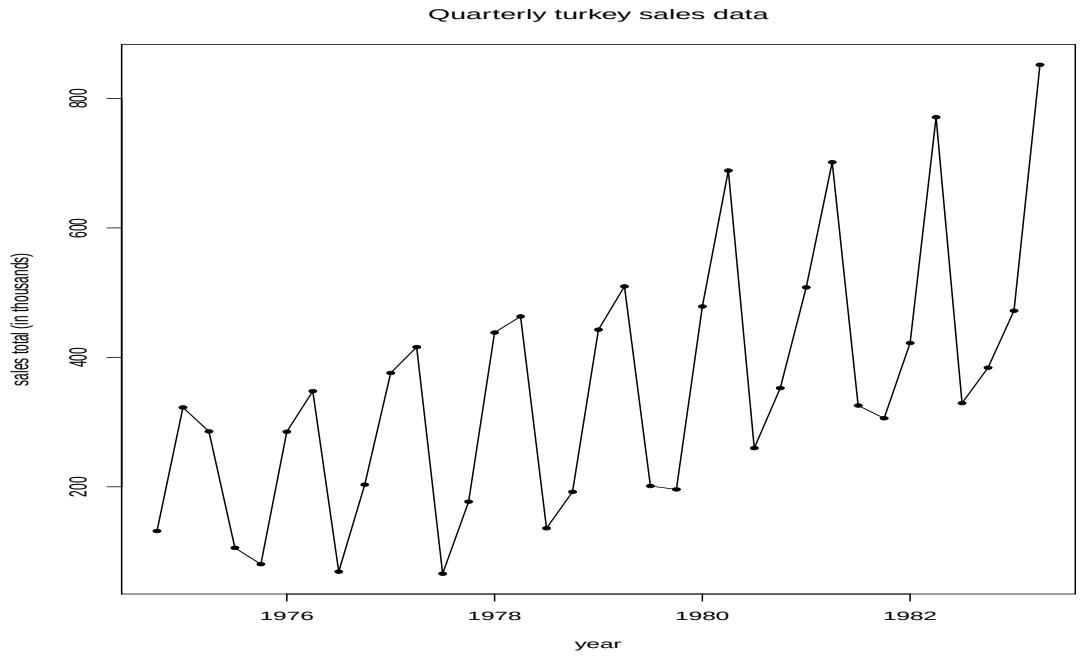


Figure 1: Quarterly sales turkey chicks data

4 The plot above shows quarterly total sales (in thousands) of one-day-old turkey chicks from hatcheries in Eire (source Pole, A., West, M., and Harrison, P.J., 1994, Applied Bayesian Forecasting and Time Series Analysis, Chapman-Hall).

(i) Briefly describe the features of the data. ***(2 marks)***

(ii) Suggest a transformation of the time series data y_t, likely to result in a stationary time series x_t and write down x_t as a function of y_t using appropriate differencing notation. ***(2 marks)***

(iii) For a time series x_t (of length 31) derived from y_t by a suitable transformation, the sample ACF and the sample PACF are tabulated below:

Lag	1	2	3	4	5	6	7	8
ACF	r_1	r_2	-0.468	0.700	-0.397	0.134	0.051	0.002

and

Lag	1	2	3	4	5	6	7	8
PACF	-0.650	-0.488	-0.832	-0.565	-0.300	0.124	0.090	0.032

(a) Find the values of r_1 and r_2. ***(4 marks)***

(b) Test whether x_t is a white noise. ***(2 marks)***

(c) Test whether x_t is consistent with autoregressive models. ***(3 marks)***

(d) Test whether x_t is consistent with moving average models. ***(5 marks)***

(e) Based on your analysis above, suggest a time series model for x_t that is likely to perform well when fitted to the data. ***(2 marks)***

5 Consider the time series model

$$y_t = 19 - \frac{1}{3}y_{t-1} - \frac{1}{4}y_{t-2} + \epsilon_t - \frac{1}{2}\epsilon_{t-1},$$

where ϵ_t is white noise with variance 8.

(i) Write down this model using the Backward shift operator B. ***(2 marks)***

(ii) Show that this model is causal and invertible. ***(5 marks)***

(iii) Find the mean of y_t. ***(3 marks)***

(iv) Find the variance of y_t. ***(10 marks)***

6 The plot below relates y_t the quarterly change in a company's sales to x_t the quarterly change of a market sales indicator variable.

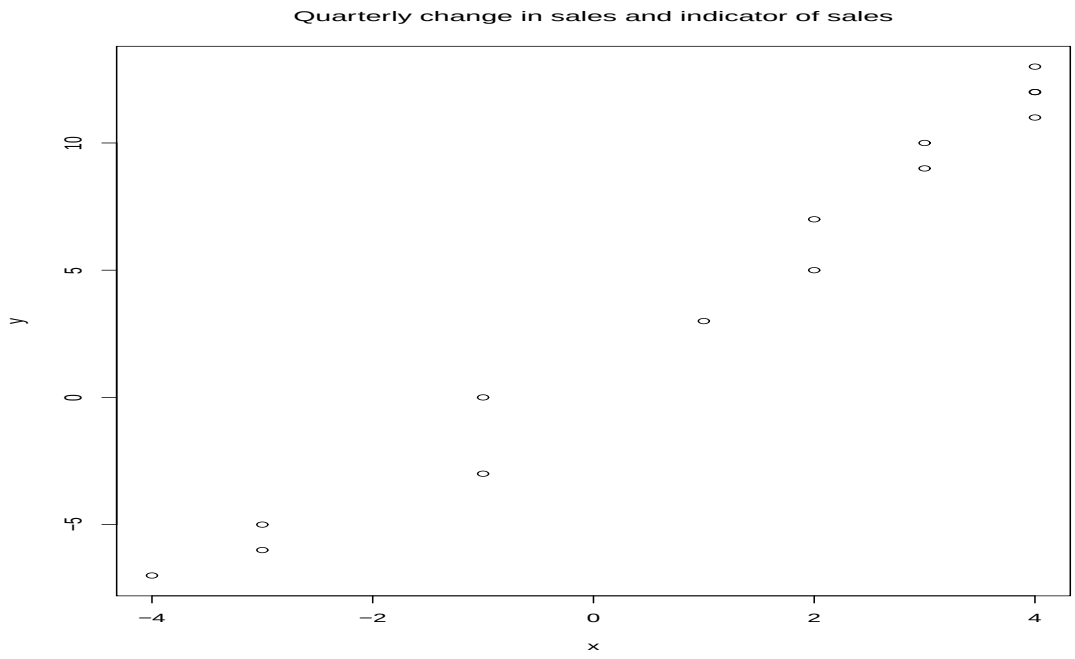


Figure 2: Relation of change of sales y_t and indicator of sales x_t

As a first model it is suggested to regress y_t on x_t, or

$$y_t = x_t\beta + \epsilon_t,$$

where β is a regression coefficient and ϵ_t is a white noise with variance 1.

However, the statistician of the company argues this model is not appropriate to model the data set. To back her argument she has provided the autocorrelation functions of the time series x_t and y_t, given in the plot overleaf (Figure 3).

(i) Explain why the statistician believes the model above is inappropriate.

(1 mark)

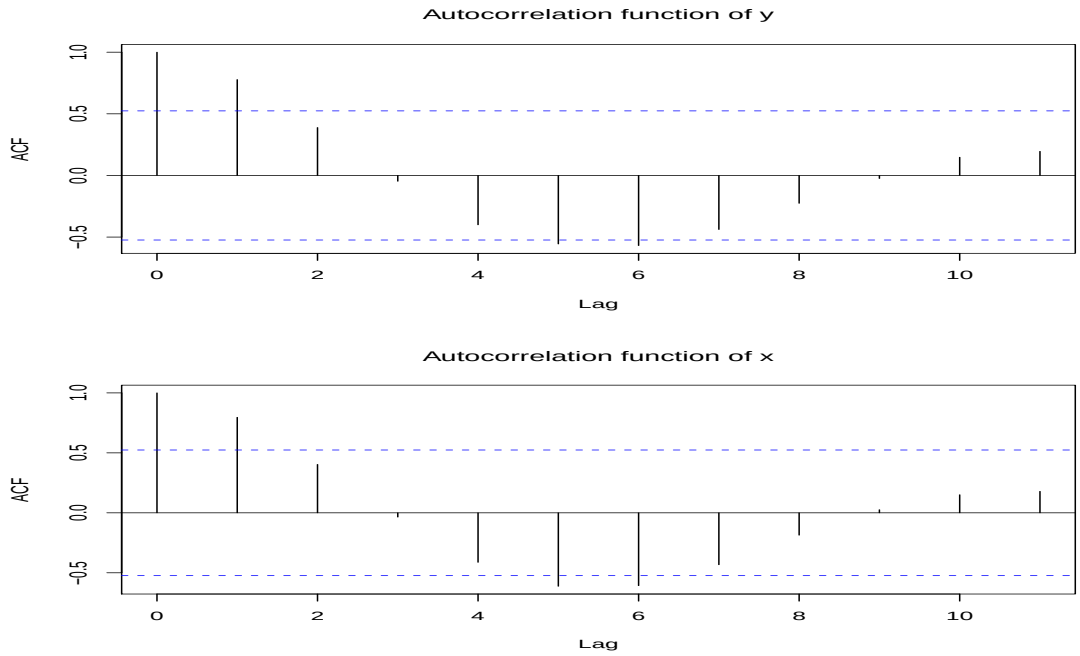


Figure 3: ACF of the time series x_t and y_t

6 (continued)

(ii) The statistician suggested a second model as alternative, given by

$$y_t = x_t\beta_t + \epsilon_t \quad \text{and} \quad \beta_t = \beta_{t-1} + \zeta_t, \tag{1}$$

where ϵ_t is as before and ζ_t is a white noise with variance 10.

(a) Give the name of model (1). ***(1 mark)***

(b) For model (1) show that $P_{t|t}$ the posterior variance of β_t satisfies

$$\frac{1}{P_{t|t}} = \frac{1}{P_{t-1|t-1} + 10} + x_t^2.$$

(7 marks)

(c) If $x_1 = 4$, $x_2 = 4$, $y_1 = 12$, $y_2 = 11$ and the prior of β_0 is $\beta_0 \sim N(2, 0.81)$, then use the result in (b) to calculate the posterior means $\hat{\beta}_{1|1}$, $\hat{\beta}_{2|2}$ and the posterior variances $P_{1|1}$, $P_{2|2}$. ***(5 marks)***

(d) If $x_3 = 3$, use (c) to obtain the one-step forecast mean of $y_3 = 9$ and the associated residual. Comment on the quality of this forecast. ***(2 marks)***

6 (continued)

(e) If instead of $\beta_0 \sim N(2, 0.81)$ the prior distribution of β_0 is set to either of the following:

(α) $\beta_0 \sim N(10, 0.81)$ or

(β) $\beta_0 \sim N(2, 100)$,

comment on whether you expect an improvement on forecasting and the general model performance for all y_t. ***(3 marks)***

(f) Suggest how the model performance can be improved. ***(1 mark)***

End of Question Paper