The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **2012–13**

Stochastic Processes and Finance **3 hours**

Candidates may bring to the examination a calculator that conforms to University regulations.
Full marks may be obtained by complete answers to five questions. All answers will be marked, but credit will be given only for the best five answers.

1 (a) If X is a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}(X^2) < \infty$ and $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra, define the conditional variance of X given $\mathcal{G}$ by

$$\text{Var}(X|\mathcal{G}) = \mathbb{E}([X - \mathbb{E}(X|\mathcal{G})]^2|\mathcal{G}).$$

Show, identifying carefully the properties of the conditional expectation that you use, that

(i) $\text{Var}(X|\mathcal{G}) = \mathbb{E}(X^2|\mathcal{G}) - \mathbb{E}(X|\mathcal{G})^2$;

(ii) $\text{Var}(X) = \mathbb{E}(\text{Var}(X|\mathcal{G})) + \text{Var}(\mathbb{E}(X|\mathcal{G}))$.

(10 marks)

(b) A fair coin is tossed twice and for each toss the uppermost face is observed to be either H, for a head, or T, for a tail, so that there are 2^2 possible outcomes each with probability $1/2^2$.

(i) Identify carefully the elements of $(\Omega, \mathcal{F}, \mathbb{P})$ in this case.

(ii) Let X_i be one if there is a head on the ith toss and zero otherwise (for $i = 1, 2$). Let $Y = X_1 + X_2$ be the number of heads. Write down the following subsets of Ω: $Y^{-1}(0)$, $Y^{-1}(1)$, $Y^{-1}(2)$. Describe fully the σ-algebra, $\sigma(Y)$, generated by Y. Explain what it means to say Z is $\sigma(Y)$-measurable.

(iii) Recall that the *conditional expectation* of X_1 given the σ-algebra $\sigma(Y)$ is defined to be the unique $\sigma(Y)$-measurable random variable $\mathbb{E}(X_1|\sigma(Y))$ for which

$$\mathbb{E}(X_1 1_A) = \mathbb{E}(\mathbb{E}(X_1|\sigma(Y))1_A),$$

for all $A \in \sigma(Y)$. Find the value of $\mathbb{E}(X_1|\sigma(Y))$ on each of the sets $Y^{-1}(0)$, $Y^{-1}(1)$ and $Y^{-1}(2)$ and in each case justify your answer by explicit calculation.

(10 marks)

2 (a) Let $(X(t), t \geq 0)$ be a stochastic process adapted to the filtration $(\mathcal{F}_t, t \geq 0)$.

(i) Explain what is meant by saying that $(\mathcal{F}_t, t \geq 0)$ is a filtration.

(ii) Explain what is meant by saying that the process is *adapted*.

(iii) Give the conditions needed for $(X(t), t \geq 0)$ to be a martingale with respect to this filtration.

(4 marks)

(b) Let $(B(t), t \geq 0)$ be a Brownian motion and $(\mathcal{F}_t, t \geq 0)$ its natural filtration.

(i) Explain what is meant by saying that $(\mathcal{F}_t, t \geq 0)$ is the Brownian motion's natural filtration.

(ii) Give the properties that define $(B(t), t \geq 0)$.

(iii) Show that $(B(t))^2 - t$ is a martingale.

(7 marks)

(c) Let $(X(t), t \geq 0)$ be an Itô process and $(B(t), t \geq 0)$ a Brownian motion .

(i) Explain what is meant by saying $(X(t), t \geq 0)$ is an Itô process with $dX(t) = G(t)dt + F(t)dB(t)$.

(ii) Use Itô's formula to give the stochastic differential of $f(X(t))$ in terms of dt and $dB(t)$ (where f has a continuous second derivative).

(iii) Give the stochastic differential of $\log(1 + (B(t))^2)$.

(iv) Give the stochastic differential of $\log(1 + (X(t))^2)$ (in terms of dt and dB).

(9 marks)

3 Throughout this question $(B(t), t \geq 0)$ is a Brownian motion and $(\mathcal{F}_t, t \geq 0)$ is its natural filtration.

(a) Let the process Y be defined by $Y(0) = 1$ and

$$dY(t) = 2tdt + 3dB(t).$$

Obtain $Y(t)$ and hence give the mean, variance and distribution of $Y(t)$.

(5 marks)

(b) (i) Give conditions on F for the stochastic integral

$$I_T(F) = \int_0^T F(t)dB(t)$$

to be defined and to have a finite variance.

(ii) Give the mean and variance of $I_T(F)$.

(iii) Give two properties of $(I_t(F), 0 \leq t \leq T)$.

(6 marks)

(c) Let Y be given by the stochastic differential equation (SDE)

$$dY(t) = dB(t) + 2tY(t)dt.$$

(i) By considering the differential of $(\exp(-t^2)Y(t))$, show that, when $Y(0) = 0$,

$$Y(t) = \int_0^t \exp(t^2 - s^2)dB(s).$$

(ii) Show that there is a unique solution to this SDE on any finite time interval $[0, T]$. (You should state any theorems from the course that you use.)

(9 marks)

4 In a finite market model there is a single risky asset with price $S(n)$ at time n, for $n = 0, 1, \dots, T$, adapted to the filtration $(\mathcal{F}_n, n = 0, 1, 2 \dots, T)$, and the interest rate is zero. Suppose that, when $S(n) = s$, $S(n+1)$ is independent of $\mathcal{F}_n$ and given by

$$S(n+1) = \begin{cases} s(1+u) & \text{with probability} \quad ap \\ s & \text{with probability} \quad 1-a \\ s(1+d) & \text{with probability} \quad a(1-p) \end{cases},$$

where $0 < p < 1$, $0 < a < 1$ and $-1 < d < 0 < u$.

(a) Explain what it means to say that this is a finite market model. In this context, what can you say about the discounted prices of the risky asset? ***(2 marks)***

(b) Calculate $\mathbb{E}[S(n+1)|\mathcal{F}_n]$. For what values of a and p is $\mathbb{E}[S(n+1)|\mathcal{F}_n] = S(n)$? ***(4 marks)***

(c) Is the market arbitrage free? Justify your answer, stating any theorems you use. ***(4 marks)***

(d) What does it mean to say that a market is complete? Is this market complete? Justify your answer, stating any theorems you use. ***(4 marks)***

(e) Suppose that a European call option is offered, maturing at time $T = 1$ and with strike price $k > S(0) = s$.

(i) Explain what this means. In what circumstances will the option be exercised? Give a formula for the value of the option at $T = 1$.

(ii) Consider hedging this option by holding ψ in cash and ϕ units of the risky asset. By comparing the value of this portfolio at time $T = 1$ with the value of the option for each of the three possible prices show that hedging is impossible. What does this imply about pricing this option?

(6 marks)

5 Let $B = (B(t), 0 \leq t \leq T)$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t, 0 \leq t \leq T)$ be its natural filtration, where $\mathcal{F}_0$ is trivial.

Suppose the market comprises a single stock $(S(t), 0 \leq t \leq T)$ and a risk-free security $(A(t), 0 \leq t \leq T)$ and that these are both Itô processes, defined by the differentials

$$dS(t) = G_S(t)dt + F_S(t)dB(t); \quad dA(t) = G_A(t)dt.$$

Define the discounted stock price by $\widetilde{S}(t) = A(t)^{-1}S(t)$.

You are given a portfolio (ϕ, ψ) where, at time t, $\phi(t)$ represents holdings in stocks and $\psi(t)$ represents holdings in the risk-free security.

(a) Write down expressions for the wealth $V(t)$ of the portfolio at time t and the discounted wealth $\widetilde{V}(t)$. What does it mean for the portfolio to be *self-financing*? Give your answer in terms of stochastic differentials.
(2 marks)

(b) Write down, and justify, the stochastic differentials for $d\widetilde{S}(t)$ in terms of $dS(t)$ and $dA(t)$. ***(4 marks)***

(c) Compute the stochastic differential of $\widetilde{V}(t)$ in terms of $dV(t)$ and $dA(t)$. When (ϕ, ψ) is self-financing, substitute for V and $dV(t)$ and rearrange to deduce that

$$d\widetilde{V}(t) = \phi(t)d\widetilde{S}(t).$$

Now show that if $d\widetilde{V}(t) = \phi(t)d\widetilde{S}(t)$ then (ϕ, ψ) is self-financing.
(8 marks)

(d) Suppose now that

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t); \quad dA(t) = r(t)A(t)dt.$$

Find the stochastic differential equation satisfied by $\widetilde{S}(t)$.

The original measure $\mathbb{P}$ is changed to the equivalent probability measure $\mathbb{Q}$ where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{\int_0^T F(s)dB(s) - \frac{1}{2}\int_0^T F(s)^2 ds\right\},$$

and Girsanov's theorem then tells us that

$$W(t) = B(t) - \int_0^t F(s)ds$$

is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q})$. Find $F = (F(t), 0 \leq t \leq T)$ for which $\widetilde{S}$ is a $\mathbb{Q}$-martingale. Give the formula for an arbitrage price at time zero for a European contingent claim $X = f(S(T))$. ***(6 marks)***

6 (a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_n, n \in \mathcal{T})$, where $\mathcal{T} = \{0, 1, \ldots, T\}$. Let $Y = (Y(n), n \in \mathcal{T})$ be an adapted integrable process and define a new process Z which is called the *Snell envelope* of Y by

$$Z(T) = Y(T)$$
$$Z(n) = \max\{Y(n), \mathbb{E}(Z(n+1)|\mathcal{F}_n)\}, n = 0, 1, \ldots, T-1.$$

(i) Let $\tau = \inf\{n \in \mathbb{Z}_+, Z(n) = Y(n)\}$. Show that τ is a bounded stopping time.

(ii) Explain what it means to say that τ solves the optimal stopping problem for $Y = (Y(n), n \in \mathcal{T})$.

(iii) Simplify $\mathbb{E}[Z(\tau \wedge (n+1))|\mathcal{F}_n]$.

(iv) Interpret τ in the context of an American contingent claim with discounted payoff $Y = (Y(n), n \in \mathcal{T})$.

(10 marks)

(b) Consider a dollar investor who wants to speculate on the future value of the euro. The price processes for a dollar cash bond and a euro cash bond are $(A(t), t \in [0, T])$ and $(D(t), t \in [0, T])$ (respectively) where

$$dA(t) = rA(t)dt; \quad dD(t) = uD(t)dt$$

so $r > 0$ and $u > 0$ are the interest rates. Let the exchange rate $E(t)$ be the value of one euro in dollars at time t. This is modelled as

$$E(t) = E_0 \exp\{\mu t + \sigma B(t)\},$$

where $\mu \in \mathbb{R}, \sigma > 0$ and B is a Brownian motion.

(i) A. Explain why $R(t) = D(t)E(t)$ corresponds to a tradeable asset for the dollar investor.

B. Obtain $dE(t)$.

C. Show that the discounted asset $\widetilde{R}(t) = A(t)^{-1}D(t)E(t)$ satisfies

$$d\widetilde{R}(t) = \sigma \widetilde{R}(t)(dB(t) - \theta dt)$$

where θ should be identified in terms of r, u, μ, and σ.

(ii) With a suitable F, Girsanov's theorem (described in the previous question) provides a measure $\mathbb{Q}$ under which $\widetilde{R}$ is a martingale. Use this to give a formula for the arbitrage price (in dollars) at time $t < T$ for the option to buy one euro bond at time T for k dollars.

(10 marks)

End of Question Paper