The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2012-13

Fields - MAS438 — **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

Blank

1 (i) For each of the subsets J_1, J_2 of $\mathbb{C}$ specified below determine, with justification, whether it is a subfield of $\mathbb{C}$:

(a) $J_1 = \{a + b\sqrt{3} : \ a, b \in \mathbb{Q}\}$, ***(5 marks)***

(b) $J_2 = \{a + b\sqrt{3} + ci : \ a, b, c \in \mathbb{Q}\}$. ***(3 marks)***

(ii) Let K and L be fields such that $[K : \mathbb{Q}] = 5$ and $[L : \mathbb{Q}] = 4$. Is the following situation possible $K \subseteq L$? Justify your answer. ***(4 marks)***

(iii) Find the subfield of $\mathbb{C}$ generated by the numbers $\{\sqrt{2}, \sqrt{3}\}$ and give a possible $\mathbb{Q}$-basis. Justify your answer. ***(9 marks)***

(iv) Let K be a subfield of a field L. Give a definition of $[L : K]$. ***(2 marks)***

(v) Express the complex number $\dfrac{(1-2i)(1+2i)}{1-i}$ in the form $a + bi$ where $a, b \in \mathbb{R}$. ***(2 marks)***

2 (i) State Gauss's Lemma. ***(3 marks)***

(ii) Prove Gauss's Lemma. ***(8 marks)***

(iii) Give the definition of a simple field extension. ***(2 marks)***

(iv) Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Prove that $K = \mathbb{Q}(b)$ where $b = \sqrt{2} + 2\sqrt{3}$. ***(8 marks)***

(v) Find $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$. ***(4 marks)***

3 (i) Let F be a finite field with q elements.

(a) Give the definition of the characteristic $p = \mathrm{char}(F)$ of the field F. Prove that p is a prime number. ***(4 marks)***

(b) Deduce that $q = p^n$ for some n. Give an expression for n. ***(5 marks)***

(c) Prove that the Frobenius map $\mathrm{Fr} : F \to F$, $a \mapsto a^p$, is an isomorphism. ***(6 marks)***

(d) State the Uniqueness Theorem for the algebraic closure $\overline{\mathbb{F}}_p$ of the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. ***(3 marks)***

(e) Prove that the field F with $q = p^n$ elements, as a subfield of $\overline{\mathbb{F}}_p$, consists precisely of the roots of the polynomial $f(x) = x^q - x \in \mathbb{F}_p[x]$. ***(7 marks)***

4 (i) Let B be a set of (at least 2) points in the plane $\mathbb{R}^2$, and $P, Q \in \mathbb{R}^2$. Explain what does it mean that the point P is **constructible in one step from** B and the point Q is **constructible from** B. ***(4 marks)***

(ii) Define the set of constructible points. ***(2 marks)***

.

(iii) Let $(a, b) \in \mathbb{R}^2$. Give a criterion for constructibility of the point (a, b) (via quadratic fields). ***(3 marks)***

(iv) Prove the criterion. ***(8 marks)***

(v) Using **only** the criterion show that the point $(\sqrt{2 + \sqrt[4]{3}}, 0)$ is a constructible point. ***(4 marks)***

(vi) Is the point $(\frac{1}{2}, \sqrt[3]{2})$ constructible? Justify your response. ***(4 marks)***

End of Question Paper