The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2012-2013**

Advanced Operations Research **2 Hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) A farm uses at least 800 kg of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions:

Feedstuff	Protein	Fibre	Cost (£ per kg)
Corn	32	27	0.3
Soybean meal	360	65	0.9

The units in the Protein and Fibre columns are gram per kg of feedstuff.

The dietary requirements of the special feed are that the diet should contain at least 25% protein and at most 6% fibre. The farm wishes to determine the daily minimum cost feed. Set up the Linear Programming model. **Do NOT try to solve it**. ***(6 marks)***

(ii) Use the Two-Phase Simplex method to solve :

$$\max \quad -2x_1 - 3x_2$$

subject to

$$2x_1 + x_2 \leq 16$$

$$x_1 + 3x_2 \geq 20$$

$$x_1 + x_2 = 10$$

$$x_1,\ x_2 \geq 0\ .$$

Determine the optimal solution (You should be able to finish the first phase by the third tableau). ***(19 marks)***

2 (i) At a given stage for a certain maximising Linear Programming Problem with the variables x_1 and x_2, the simplex tableau in standard form is

	x_1	x_2	x_3	x_4	Solution
z	0	1/2	1/2	0	3/2
x_1	1	1/2	−1/2	0	3/2
x_4	0	−3/2	−1/2	1	−3/2

Establish the optimality and feasibility of this tableau. Using the Dual Simplex Algorithm, complete *one* further tableau and comment upon the optimality and feasibility of this tableau. ***(6 marks)***

(ii) Write down the Complementary Slackness conditions for the primal and dual Linear Programming problems

$$\begin{array}{llll} \max & z = \mathbf{c}^{\mathsf{T}}\mathbf{x}\ , & \mathbf{A}\mathbf{x} \leq \mathbf{b}\ , & \mathbf{x} \geq \mathbf{0} \\ \min & w = \mathbf{b}^{\mathsf{T}}\mathbf{y}\ , & \mathbf{A}^{\mathsf{T}}\mathbf{y} \geq \mathbf{c}\ , & \mathbf{y} \geq \mathbf{0} \end{array}$$

and interpret their meaning. ***(5 marks)***

(iii) Find the dual of the following problem using its canonical form:

$$\min z = x_1 + x_2 + 3x_3$$

such that

$$x_1 - x_2 + x_3 \leq 3$$

$$2x_1 + x_2 + x_3 \geq 2$$

$$x_1 + x_2 - x_3 \leq 5$$

$$x_1,\ x_2,\ x_3 \geq 0$$

(6 marks)

(iv) A university is in the process of forming a committee to handle students' grievances, which must include at least one female, one male, one student, one administrator, and one staff member. Also, there cannot be more staff members than students. Ten individuals (identified by letters a to j) have been nominated, in the categories given as follows:

Category	Individuals
Females	a, b, c, d, e
Males	f, g, h, i, j
Students	a, b, c, j
Administrators	e, f
Staff	d, g, h, i

The university wants to form the smallest committee with representation from each of the five categories. Formulate the problem as an integer programming problem. Do NOT try to solve it. ***(8 marks)***

3 (i) Consider the following 2×4 two-person zero-sum game played by players A and B. The payoff matrix from player A's point of view is

$$\begin{bmatrix} -2 & -5 \\ -3 & -4 \\ -3 & -2 \\ 1 & -6 \end{bmatrix}$$

Show that the game has no pure strategy equilibrium solution. Find the mixed strategy equilibrium solution. ***(12 marks)***

(ii) Beyonce's agent has contacted you to plan the positioning of song tracks on the CD version of her next double CD album. The length and type of the songs are tabulated in the table below. The function to be minimised is the difference in length between the total track times on CD1 and CD2.

The assignment of songs to the CD's must satisfy the following conditions:

(a) Each CD must have exactly two ballads.

(b) CD 1 must have at least three hit songs.

(c) If songs 2 and 4 are on CD 1, then song 5 must be on CD 2.

Song	Type	Length (in minutes)
1	Ballad	4
2	Hit	5
3	Ballad	3
4	Hit	2
5	Ballad	4
6	Hit	3
7	Unspecified	5
8	Ballad and hit	4

Define suitable variables and write down constraints so that the problem can be solved using integer programming. **DO NOT SOLVE THE RESULTING PROBLEM.** ***(11 marks)***

Describe in words **briefly** how you may use the Branch and Bound algorithm to solve the problem. ***(2 marks)***

4 (i) Consider the following linear programming problem

$$\max\ z = c^T x$$

subject to

$$Ax = b$$

where A is a $m \times n$ matrix. Let B be the basis matrix, and $z_j = c_B^T B^{-1} a_j$, where a_j is the jth column of A, and c_B is the vector of cost coefficients corresponding to B. We assume B is composed of the first m columns of A.

(a) Show that $z_j - c_j \geq 0$ when B is the optimal basis.

(b) Show that for any feasible solution $\widetilde{x}$, we have

$$\widetilde{x}_B + \sum_{j=m+1}^{n} \widetilde{x}_j B^{-1} a_j = B^{-1} b\,,$$

where $\widetilde{x}_B = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_m)^T$. ***(15 marks)***

(ii) Given the primal linear programming problem: $\max z = c^T x$ subject to $Ax \leq b$ and $x \geq 0$, define suitable dual variables and write down its Lagrangian dual function. ***(4 marks)***

Use the Lagrangian dual function to show that the dual problem is $\min v = b^T y$ subject to $y^T A \geq c^T$ and $y \geq 0$. ***(6 marks)***

5 A factory manufactures four types of table, namely 1, 2, 3 and 4. Manufacturing one unit of each type of table requires 1, 3, 8 and 4 hours of machine time, and 1, 1, 1 and 3 worker days, respectively. 90 hours of machine time and 80 worker days per week are available. The profits per table are respectively 1, 2, 4 and 3 units. The problem is solved using the simplex method and the optimal tableau is

	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	0.5	0.5	0.5	0.5	85
x_2	0	1	3.5	0.5	0.5	−0.5	5
x_1	1	0	−2.5	2.5	−0.5	1.5	75

with the variables x_i being the number of type i tables, and the slack variables x_5 and x_6 relating to the machine time and worker day constraints respectively.

(i) Write down the optimal solution and indicate which constraints are binding. ***(3 marks)***

(ii) Determine the range of allowable variations in the unit profit value of type 1 tables which would leave the optimal basis unchanged. ***(7 marks)***

(iii) Write down the values of the optimal dual variables and interpret their meaning. ***(2 marks)***

(iv) Determine the range of allowable variations in the availability of the machine time which would leave the optimal basis unchanged. ***(6 marks)***

(v) The available machine time per week could be increased by 5 hours at a total cost of 1 unit. Establish whether this action would be worthwhile. ***(3 marks)***

(vi) An additional type of table can be produced. To manufacture one of these tables requires 1 hour of machine time and 2 worker days. The profit per table is 3 units. Establish whether the previous basis remains optimal. ***(4 marks)***

End of Question Paper