The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012–2013**

MAS350 Measure and Probability **2 hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Let S be a set. Give precise definitions of

(a) A *σ-algebra* Σ of subsets of S. ***(3 marks)***

(b) A *finite measure* on the measurable space (S, Σ). ***(3 marks)***

(ii) If Σ is a σ-algebra of subsets of S and (A_n) is a sequence of subsets of Σ, explain why $\bigcap_{n=1}^{\infty} A_n \in \Sigma$. ***(2 marks)***

(iii) Suppose that m is a finite measure on the measurable space (S, Σ) and (A_n) is an increasing sequence of subsets of Σ, so that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$. Prove that

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n).$$

Hence deduce that if (B_n) is a decreasing sequence of subsets of Σ, so that $B_{n+1} \subseteq B_n$ for all $n \in \mathbb{N}$, then

$$m\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} m(B_n).$$

(10 marks)

(iv) We construct a variant on the Cantor set as follows. Start with the interval $[0, 1]$ and remove the middle $1/7$ to obtain the set D_1. Then remove the middle $1/7$ of each of the disjoint intervals comprising D_1 to obtain D_2. Iterate this procedure to obtain a sequence of sets (D_n) and define $D = \bigcap_{n=1}^{\infty} D_n$. Deduce a formula for the Lebesgue measure of D_n (there is no need to formally prove this) and hence obtain the Lebesgue measure of D, stating clearly any results you use to justify this last deduction. ***(7 marks)***

2 Throughout this question (S, Σ, m) is a measure space and $\mathbb{R}$ is equipped with its usual Borel σ-algebra.

(i) Give two equivalent formulations of what it means for a function $f : S \to \mathbb{R}$ to be measurable. ***(2 marks)***

(ii) Let $f : S \to \mathbb{R}$ be the *indicator function* $\mathbf{1}_A$ where $A \in \Sigma$. Give a precise definition of this function and explain carefully why it is measurable. ***(5 marks)***

(iii) Let f and g be measurable functions from S to $\mathbb{R}$ and let c and d be real numbers.

(a) Show that $c - g$ is a measurable function, where $(c - g)(x) = c - g(x)$ for all $x \in S$. ***(4 marks)***

(b) Show that $B \in \Sigma$, where $B = \{x \in S; f(x) < g(x)\}$. ***(4 marks)***

(c) Use the result of (b) to show that $f + g$ is measurable. ***(3 marks)***

(d) Deduce that $cf + dg$ is measurable. ***(3 marks)***

(iv) Explain briefly why each of the following functions $f : \mathbb{R} \to \mathbb{R}$ are measurable.

(a) $f(x) = \sin(x)$,

(b) $f(x) = \sin(x) + \mathbf{1}_{[0,1]}(x)$,

(c) $f(x) = \mathbf{1}_{[0,1]}(x) \sin(x)$,

(d) $f(x) = \sin(\mathbf{1}_{[0,1]}(x))$. ***(4 marks)***

3 Throughout this question (S, Σ, m) is a measure space and $\mathbb{R}$ is equipped with its usual Borel σ-algebra.

(i) (a) Explain how to define $\int_S f dm$ in the case where $f : S \to \mathbb{R}$ is a non-negative *simple* function, i.e. $f = \sum_{i=1}^{n} c_i \mathbf{1}_{A_i}$ for some $n \in \mathbb{N}$, where $c_1, \ldots, c_n \in [0, \infty)$ and $A_1, \ldots A_n \in \Sigma$ are mutually disjoint with $S = \bigcup_{i=1}^{n} A_i$. ***(2 marks)***

(b) Explain how to extend the definition of $\int_S f dm$ to the case where $f : S \to \mathbb{R}$ is an arbitrary non-negative measurable function. What does it mean for f to be *integrable*? ***(3 marks)***

(c) If f is as in (b) and $c > 0$, prove that $\int_S cf dm = c \int_S f dm$. ***(5 marks)***

(d) If f and g are as in (b) with $f \leq g$ show that $\int_S f dm \leq \int_S g dm$. ***(2 marks)***

(ii) (a) Explain why the mapping $x \to e^{-x}$ is integrable on $[0, \infty)$, with respect to Lebesgue measure, stating clearly any results that you use. ***(4 marks)***

(b) Deduce that $x \to \dfrac{e^{-x}}{1 + x^2}$ is integrable on $[0, \infty)$, with respect to Lebesgue measure. ***(2 marks)***

(iii) Let $f : S \to \mathbb{R}$ be integrable and define $\mu = \int_S f dm$. Prove the following version of *Chebychev's inequality*:

$$m(\{x \in S; |f(x) - \mu| \geq c\}) \leq \frac{1}{c^2} \int_S |f - \mu|^2 dm,$$

where $c > 0$. Write down a probabilistic interpretation of this result using the concept of variance in the case where f is a random variable, which we denote by X, defined on a probability space $(\Omega, \mathcal{F}, P)$. ***(7 marks)***

4 Throughout this question $\mathbb{R}$ is equipped with its usual Borel σ-algebra.

(i) Let (S, Σ, m) be a measure space. State the *Lebesgue dominated convergence theorem* in the context of the integration of real-valued measurable functions defined on S. ***(3 marks)***

(ii) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is integrable. Deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{n}{x^2+n} f(x)dx = \int_{\mathbb{R}} f(x)dx.$$

Hence, or otherwise, show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{x^2}{x^2+n} f(x)dx = 0.$$

(5 marks)

(iii) (a) Let (S, Σ, m) be a measure space and $f : [a,b] \times S \to \mathbb{R}$ be a measurable function for which

(I) The mapping $x \to f(t,x)$ is integrable for all $t \in [a,b]$,

(II) The mapping $t \to f(t,x)$ is differentiable for all $x \in S$,

(III) There exists a non-negative integrable function $h : S \to \mathbb{R}$ so that $\left|\frac{\partial f(t,x)}{\partial t}\right| \leq h(x)$ for all $t \in [a,b], x \in S$.

Show that the mapping $t \to \int_S f(t,x)dm(x)$ is differentiable on (a,b) and that

$$\frac{d}{dt}\int_S f(t,x)dm(x) = \int_S \frac{\partial f(t,x)}{\partial t} dm(x).$$

(6 marks)

(b) Let $f : [1,\infty) \to \mathbb{R}$ be an integrable function such that $x \to \frac{f(x)}{x}$ is also integrable. Deduce that the mapping $t \to \int_{[1,\infty)} \cos(tx)\frac{f(x)}{x}dx$ is differentiable and find its derivative. ***(6 marks)***

(iv) Let (S, Σ, m) be a finite measure space and $f : S \to \mathbb{R}$ be measurable. Show that if $|f|^n$ is integrable for some $n \in \mathbb{N}$ with $n > 1$ then $|f|^r$ is integrable for all $1 \leq r \leq n$. [Hint: Write $|f| = |f|\mathbf{1}_{\{f\leq 1\}} + |f|\mathbf{1}_{\{f>1\}}$.] ***(5 marks)***

5 Let $(\Omega, \mathcal{F}, P)$ be a probability space and (A_n) be a sequence of events so that $A_n \in \mathcal{F}$ for all $n \in \mathbb{N}$.

(i) (a) Explain precisely what it means for the events (A_n) to be *independent.* ***(2 marks)***

(b) Define the events $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$. ***(2 marks)***

(c) Show that $\liminf_{n\to\infty} A_n = \left(\limsup_{n\to\infty} A_n^c\right)^c$ and hence deduce that

$$P\left(\liminf_{n\to\infty} A_n\right) = 1 - P\left(\limsup_{n\to\infty} A_n^c\right).$$

(5 marks)

(d) State *Kolmogorov's* $0-1$ *law.* What does this tell us about the possible values of $P\left(\limsup_{n\to\infty} A_n\right)$? ***(3 marks)***

(ii) (a) The first part of the *Borel-Cantelli lemma* states that if

$$\sum_{n=1}^{\infty} P(A_n) < \infty, \text{ then } P\left(\limsup_{n\to\infty} A_n\right) = 0.$$

Prove this result. ***(4 marks)***

(b) The second part of the *Borel-Cantelli lemma* deals with *independent events.* Write down the statement of this result (a proof is not required.) ***(2 marks)***

(iii) Consider a biased coin for which heads has probability 0.52. The coin is tossed repeatedly and successive tosses are regarded as independent events. Show that the sequence $THHHHT$ occurs infinitely often (where H and T stand for "head" and "tail", respectively). ***(7 marks)***

End of Question Paper