The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012–2013**

MAS346 Groups and Symmetry **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Let X be a set. Prove that the set $S(X)$ of all bijections $f : X \to X$ is a group under composition of functions. ***(6 marks)***

(ii) Let G be a group.

(a) Prove that for $a \in G$ the map $l_a : G \to G$ defined by $x \mapsto l_a(x) := ax$ is an element of $S(G)$. ***(5 marks)***

(b) Give an example to show that l_a is in general not an automorphism of G. ***(2 marks)***

(iii) For G a group and $a \in G$ let $\omega_a : G \to G$ be defined by $x \mapsto \omega_a(x) := axa^{-1}$. The group of inner automorphisms of G is defined to be

$$\mathrm{Inn}(G) = \{\omega_a : a \in G\}.$$

(a) Prove that $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$.
[You may assume that $\mathrm{Inn}(G) < \mathrm{Aut}(G)$.] ***(4 marks)***

(b) Determine $\mathrm{Aut}(\mathbf{Z}/5\mathbf{Z})$ and $\mathrm{Inn}(\mathbf{Z}/5\mathbf{Z})$ explaining your reasoning. ***(4 marks)***

(c) Determine $\mathrm{Inn}(S_3)$ using that $Z(S_3) = \{e\}$. By considering how many elements $\mathrm{Aut}(S_3)$ can maximally contain deduce that $\mathrm{Inn}(S_3) = \mathrm{Aut}(S_3)$. ***(4 marks)***

2 (i) Define the centre of a group and prove that it is a normal subgroup. ***(4 marks)***

(ii) Find the centre of the group $\mathrm{GL}_3(\mathbf{R})$. ***(7 marks)***

(iii) Let H be the set of all 3×3 real matrices of the form

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix} \text{ with } a, b, c, d \in \mathbf{R} \text{ and } ad - bc, e \in \mathbf{R}^*.$$

(a) Show that H is a subgroup of $\mathrm{GL}_3(\mathbf{R})$. ***(6 marks)***

(b) Find the centre $Z(H)$ of H. ***(4 marks)***

(iv) Given any two groups G_1 and G_2 prove that the centre of the direct product $Z(G_1 \times G_2)$ is given by the direct product of the centres $Z(G_1) \times Z(G_2)$. ***(4 marks)***

3 (i) (a) Define the group O_2, and the elements R_θ and S_θ of O_2. ***(3 marks)***

(b) Put $H_n = \{A \in O_2 | A^n = 1\}$. Show that if n is odd then H_n is a finite subgroup of O_2 and state which standard group it is isomorphic to. ***(6 marks)***

(ii) Prove that the symmetry group of a regular tetrahedron centered at the origin is isomorphic to S_4. ***(6 marks)***

(iii) (a) Give the definition of a simple group. ***(2 marks)***

(b) Let G be a finite simple group. Let p be a prime dividing the order of G, and let X be a set of order n on which the group acts. Suppose that the action is nontrivial, so there is an element $g \in G$ and an element $x \in X$ such that $gx \neq x$. Use the group action to define a homomorphism $G \to S_n$ and prove that it is injective. Deduce that we must have $n \geq p$. ***(8 marks)***

4 (i) State the Sylow theorems. You should carefully define all the terms and notation used. ***(5 marks)***

(ii) Let G be a finite group of order pqr, where p, q and r are distinct primes with $p < q < r$. Prove that G has a normal Sylow subgroup of order either p, q or r. ***(10 marks)***

(iii) Let G be a group of order 60, and assume that G has a cyclic normal subgroup N of order 12. Let P be a Sylow 5-subgroup of G.

(a) Express $\mathrm{Aut}(N)$ as a product of cyclic groups of prime power order. ***(3 marks)***

(b) Show that every homomorphism $\phi : P \to \mathrm{Aut}(N)$ is trivial. ***(1 mark)***

(c) Deduce that every element of P commutes with every element of N, and explain why this implies that G is isomorphic to $P \times N$. ***(6 marks)***

End of Question Paper