The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2012-2013**

Applied Differential Equations **2 hours**

Attempt all FOUR questions.

1 For the equation $y'(x) = f(x, y(x))$, the AB3 method is defined as

$$y_{n+1} = y_n + \frac{1}{12}h(23f_n - 16f_{n-1} + 5f_{n-2}),$$

and the AM2 method is defined as

$$y_{n+1} = y_n + \frac{1}{12}h(5f_{n+1} + 8f_n - f_{n-1}),$$

where $f_n = f(x_n, y_n)$ according to the usual notations. For the AB3 method, the local truncation error is $T^P = \frac{3}{8}h^4 y^{(4)}(\xi_1)$, while for AM2 it is $T^C = -\frac{1}{24}h^4 y^{(4)}(\xi_2)$, where $x_{n-2} \le \xi_1 \le x_{n+1}$ and $x_{n-1} \le \xi_2 \le x_{n+1}$.

(i) Given the differential equation and initial condition

$$y'(x) = -3y^2 \sin(x), \quad y(0) = 0.5, \tag{1}$$

and the values $y(0.1) = 0.4963$ and $y(0.2) = 0.4855$, apply the ABM method (with AB3 as the predictor and AM2 as the corrector) to find the approximate solution at $x = 0.3$, using step size $h = 0.1$. Work throughout correct to four decimal places (note the argument of $\sin(x)$ should be understood in radians). ***(8 marks)***

(ii) Estimate the local truncation error for the approximate solution at $x = 0.3$ using Milne's device. ***(10 marks)***

(iii) Show that the AM2 method is convergent. ***(7 marks)***

2 A single step method for equation $y' = f(x, y)$ is defined by the following formulas:

$$k_1 = hf_n, \quad k_2 = hf\left(x_n + \frac{5}{6}h, y_n + \frac{5}{6}k_1\right),$$
$$y_{n+1} = y_n + \frac{2}{5}k_1 + \frac{3}{5}k_2\ ,$$

where $f_n = f(x_n, y_n)$.

(i) Find the interval of absolute stability for the method when it is applied to the test equation $y' = \lambda y$, where λ is a constant. ***(10 marks)***

(ii) Show that the method is at least of order 2. You may use the formula for the Taylor expansion of a function $f(x, y)$:

$$f(x+h, y+k) = f(x, y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{2}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + ...$$

where h and k are constants. ***(15 marks)***

3 Laplace's equation for the function $u(r, \theta)$ in polar coordinates (r, θ) is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

(i) Considering a separable solution $u(r, \theta) = R(r)\Theta(\theta)$, show that

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \alpha\ .$$

Explain why α must be a constant. ***(6 marks)***

(ii) Given that there are only trivial solutions when $\alpha \leq 0$, apply the periodic boundary conditions to find α, hence find the solution to $u(r, \theta)$ for $r > a$, subject to the conditions

$$\left.\frac{\partial u(r,\theta)}{\partial r}\right|_{r=a} = u_0 \cos\theta, \quad \text{and} \quad u(r,\theta) \to 0, \text{ when } r \to \infty.$$

(19 marks)

4 Find the solution $u(x, y)$ to the following second order hyperbolic partial differential equation

$$2\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 2,$$

subject to conditions

$$u(x, 0) = e^{-x^2}, \quad \left.\frac{\partial u(x,y)}{\partial y}\right|_{y=0} = 0\ , \quad (-\infty < x < +\infty).$$

Hint: start with suitable variable substitutions. ***(25 marks)***

End of Question Paper