The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012–2013**

Mathematical Methods **2 hours**

Marks will be awarded for your best ***FOUR*** *answers. The marks awarded to each question or section of question are shown in italics.*

1 The Fourier transform, $\hat{f}(k)$, of a function $f(x)$ is defined by

$$\hat{f}(k) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx.$$

(a) Using the above definition, derive the result

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\,\hat{f}(k)\, dk.$$ ***(7 marks)***

$$\left[\text{ You may assume that } \quad \int_{-\infty}^{\infty} e^{ikx}\, dk = 2\pi\delta(x). \right]$$

(b) Show that the Fourier transform of

$$f(x) = e^{-|x|}$$

is given by

$$\hat{f}(k) = \frac{2}{1+k^2}.$$ ***(6 marks)***

By applying the inverse Fourier transform to $\hat{f}(k)$, deduce that for real x

$$\int_0^{\infty} \frac{\cos kx}{1+k^2}\, dk = \frac{\pi}{2}e^{-|x|}.$$ ***(7 marks)***

Verify that this is correct for $x = 0$. ***(5 marks)***

2 (a) The function $x(t)$ satisfies the ordinary differential equation

$$\ddot{x} - 5\dot{x} + 6x = e^t$$

for $t > 0$, with $x(0) = \dot{x}(0) = 0$.

By taking the Laplace transform of the equation, show that

$$\tilde{x}(s) = \frac{1}{(s-1)(s-2)(s-3)},$$

where $\tilde{x}(s)$ is defined by $\quad \tilde{x}(s) = \int_0^\infty e^{-st}\, x(t)\, dt.$ ***(6 marks)***

Hence find the solution $x(t)$ for $t > 0$. ***(6 marks)***

[You may assume that the following hold:

$$\mathcal{L}\left\{x^{(n)}(t)\right\} = s^n \tilde{x}(s) - s^{n-1}x(0) - s^{n-2}\dot{x}(0) - \cdots - x^{(n-1)}(0)$$

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a} \qquad \text{for} \quad \operatorname{Re} s > a,$$

where $\mathcal{L}\{\cdot\}$ denotes the Laplace transform.]

(b) $x(t)$ satisfies

$$\ddot{x} - 5\dot{x} + 6x = f(t)$$

for $t > 0$ for some function $f(t)$, with $x(0) = \dot{x}(0) = 0$.

Show that

$$\tilde{x}(s) = \tilde{f}(s)\left(\frac{1}{s-3} - \frac{1}{s-2}\right).$$ ***(3 marks)***

Hence derive the solution

$$x(t) = \int_0^t f(u)\left\{e^{3(t-u)} - e^{2(t-u)}\right\}\, du \qquad (1)$$

for $t > 0$. ***(5 marks)***

[You may assume that the convolution theorem holds:

$$\mathcal{L}\left\{\int_0^t f(u)\, g(t-u)\, du\right\} = \tilde{f}(s)\tilde{g}(s).$$]

(c) Use equation (1) to find the solution $x(t)$ for $t > 0$ when $f(t) = e^{3t}$. ***(5 marks)***

3 The function $y(x)$ satisfies the ordinary differential equation

$$y'' + y' - 6y = \ln x \qquad 1 \le x < \infty, \tag{2}$$

with the boundary conditions

$$\begin{aligned} &y = 0 \quad \text{at} \quad x = 1 \\ &y \quad \text{finite as} \quad x \to \infty. \end{aligned} \tag{3}$$

(a) Find the independent solutions of

$$y'' + y' - 6y = 0.$$ ***(3 marks)***

(b) Given that Green's function $G(x;\xi)$ for (2) and (3) is continuous at $x = \xi$, and that $\partial G/\partial x$ has a discontinuity of size 1 at $x = \xi$, show that

$$G(x;\xi) = \begin{cases} \dfrac{1}{5} e^{-2\xi} \left(e^{5-3x} - e^{2x}\right) & 1 \leqslant x < \xi \\ \dfrac{1}{5} \left(e^{5-2\xi} - e^{3\xi}\right) e^{-3x} & \xi < x < \infty. \end{cases}$$ ***(16 marks)***

(c) Using Green's function to find the solution $y(x)$ to (2) and (3), show that

$$y'(1) = -e^2 \int_1^\infty e^{-2\xi} \ln \xi \, d\xi.$$ ***(6 marks)***

4 Consider the equation

$$\epsilon x^3 - x^2 + 4 = 0, \tag{4}$$

where ϵ is a constant satisfying $0 < \epsilon \ll 1$.

(a) The solution can be written as

$$x = \frac{1}{\epsilon}\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots\right),$$

where $x_0, x_1, x_2, \ldots$ are $O(1)$ as $\epsilon \to 0$.

Substitute into equation (4), and by considering terms of $O\left(\epsilon^n\right)$ for $n = -2, -1, 0, 1$, derive the solutions

$$x \sim \begin{cases} \dfrac{1}{\epsilon} - 4\epsilon + O\left(\epsilon^3\right) \\ 2 + 2\epsilon + O\left(\epsilon^2\right) \\ -2 + 2\epsilon + O\left(\epsilon^2\right), \end{cases}$$

as $\epsilon \to 0$. ***(19 marks)***

(b) Given the rearrangement

$$x = \left(4 + \epsilon x^3\right)^{1/2}$$

of (4), use iteration to find the solution close to 2, correct to $O\left(\epsilon^2\right)$ as $\epsilon \to 0$. ***(6 marks)***

5 The *exponential integral* is defined by

$$E(x) = \int_1^\infty t^{-1} e^{-xt}\, dt \qquad \text{for} \qquad x > 0.$$

(a) Show, by changing variables, that

$$e^x E(x) = \int_0^\infty \frac{e^{-xv}}{1+v}\, dv.$$ ***(3 marks)***

Use the sum of a geometric progression to show that

$$\frac{1}{1+v} = 1 - v + v^2 - v^3 + \cdots + (-v)^{n-1} + \frac{(-v)^n}{1+v}.$$ ***(3 marks)***

By using the above results and considering

$$I_n(x) = \int_0^\infty v^n\, e^{-xv}\, dv \qquad \text{for} \qquad n = 0, 1, 2, \ldots$$

deduce that

$$e^x E(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \cdots + \frac{(-1)^{n-1}(n-1)!}{x^n} + R_n(x),$$

where

$$R_n(x) = (-1)^n \int_0^\infty \frac{v^n\, e^{-xv}}{1+v}\, dv.$$ ***(11 marks)***

(b) By considering

$$\left| \frac{R_n(x)}{\frac{(-1)^{n-1}(n-1)!}{x^n}} \right|$$

as $x \to \infty$, show that $E(x)$ has the asymptotic series

$$E(x) \sim e^{-x} \left(\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \cdots + \frac{(-1)^n\, n!}{x^{n+1}} + \cdots \right)$$

as $x \to \infty$. ***(8 marks)***

End of Question Paper