The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012–2013**

Fluid Mechanics I **2 hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 Consider a two-dimensional incompressible flow of an inviscid fluid, where the vorticity ω takes a constant value ω_0 inside a circle of radius a, and ω is 0 outside.

(i) Assume that the flow is axisymmetric, that is, in polar coordinates (r, θ) nothing depends on θ. Write down the vorticity-stream function relationship

$$\nabla^2 \psi = -\omega \tag{1}$$

explicitly for this flow. You may use the following formulae in polar coordinates (r, θ):

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \; u_\theta = -\frac{\partial \psi}{\partial r},$$

and

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

(5 marks)

(ii) Solve the above equation (1) for ψ both in $r \leq a$ and in $r > a$. Your answer should contain a total of four integration constants.

(5 marks)

(iii) Assume that there are no singularities in the flow. Determine the integration constants by using the two continuity conditions at $r = a$, one for ψ and the other for the circumferential velocity u_θ. Write down ψ explicitly.

(10 marks)

(iv) Deduce the velocity $\mathbf{u} = (u_r, u_\theta)$ explicitly and give a physical interpretation of its profile in each region. ***(5 marks)***

2 As a model for a boundary layer, we consider an ordinary differential equation

$$\epsilon \frac{d^2u}{dx^2} + \frac{du}{dx} = 1, \qquad (1)$$

where ϵ (> 0) is a small parameter. The boundary conditions are

$$u(0) = 0 \text{ and } u(1) = 2.$$

(i) By solving the above equation, show that the exact solution is given by

$$u_{\text{exact}}(x) = x + \frac{1 - \exp(-x/\epsilon)}{1 - \exp(-1/\epsilon)}.$$

Sketch this solution. ***(11 marks)***

(ii) Put $\epsilon = 0$ in (1) and solve the resultant equation with a boundary condition $u(1) = 2$. Give a reason why we can no longer satisfy the other boundary condition. ***(4 marks)***

(iii) Give an approximate form $u_{\text{out}}(x)$ to the solution $u_{\text{exact}}(x)$ by assuming $x \gg \epsilon$. State whether this agrees with the solution obtained in (ii) or not. ***(4 marks)***

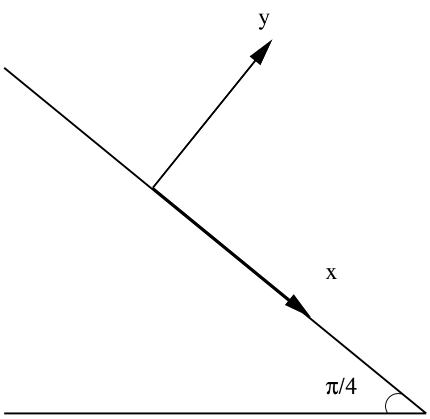
(iv) Give an approximate form $u_{\text{in}}(x)$ to the solution $u_{\text{exact}}(x)$ by assuming $x \ll \epsilon$. ***(3 marks)***

(v) Confirm that

$$\lim_{x \to 0} u_{\text{out}}(x) = \lim_{x/\epsilon \to \infty} u_{\text{in}}(x).$$

(3 marks)

Note: The symbols $\gg$ and $\ll$ mean "much larger than" and "much smaller than", respectively.



3 An incompressible viscous fluid of uniform density ρ flows steadily under gravity down a plane inclined at an angle of $\pi/4$ to the horizontal. The fluid layer is of finite thickness h and has a free surface (i.e. stress-free surface)

(i) Using the co-ordinate system shown in the figure, derive the x- and y-components of the momentum equations. Show that the pressure p is given by

$$p = -\frac{\rho g y}{\sqrt{2}} + f(x),$$

where $f(x)$ is an arbitrary function of x. You may assume that $\dfrac{\partial u}{\partial x} = 0$, where u is the component of velocity in the x-direction.

(9 marks)

(ii) State the boundary conditions on the free surface at $y = h$. ***(3 marks)***

(iii) Given that the atmospheric pressure p_0 is a constant show that

$$p - p_0 = \frac{\rho g(h-y)}{\sqrt{2}}.$$

Assuming that the thickness of the fluid layer h is uniform, show also that the profile of the component of the velocity in the x-direction is given by

$$u = \frac{g}{2\sqrt{2}\nu} y(2h - y),$$

where ν is the kinematic viscosity. ***(9 marks)***

(iv) Calculate the volume flux down the plane (per unit length across any fixed plane perpendicular to the motion). ***(4 marks)***

4 An incompressible, viscous fluid occupies the region defined by $0 < y < \infty$ above a plane rigid boundary along $y = 0$, which oscillates to and fro in the $x-$direction with speed $U \cos \omega t$. The fluid is at rest far from the boundary. Body forces can be neglected.

(i) Show that the velocity field $\mathbf{v} = (u, v, w)$ is such that $v = w = 0$ and u satisfies

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2},$$

where ν is the kinematic viscosity. ***(9 marks)***

(ii) By seeking a solution of the form

$$u = \text{Re}[f(y)e^{i\omega t}],$$

where Re denotes the real part, show that

$$u(y, t) = Ue^{-\eta} \cos(\omega t - \eta),$$

where

$$\eta = \left(\frac{\omega}{2\nu}\right)^{1/2} y.$$

[Hint: $(1 + i)^2 = 2i$.] ***(8 marks)***

(iii) Show that the shearing stress on the rigid boundary can be expressed as

$$U\sqrt{\mu\rho\omega} \cos\left(\omega t + \frac{5\pi}{4}\right),$$

where $\mu (= \nu\rho)$ is the dynamic viscosity.

[Hint: $\cos(a + b) = \cos a \cos b - \sin a \sin b$.] ***(8 marks)***

5 A solid sphere of radius a and center O is moving with constant velocity $\mathbf{V}$ in a viscous incompressible fluid. In terms of spherical polar coordinates (r, θ, ϕ), the stream-function is given by

$$\psi = V f(r) \sin^2 \theta,$$

where

$$f(r) = \frac{3ar}{4} - \frac{a^3}{4r}$$

and $V = |\mathbf{V}|$. In this problem, $u_\phi = 0$.

(i) Verify that at the surface of the sphere the conditions

$$u_r = V \cos\theta, \ u_\theta = -V \sin\theta,$$

are satisfied.

You may use the following formulae relating velocity and stream function in spherical polar coordinates:

$$u_r = \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}, \ u_\theta = \frac{-1}{r \sin\theta} \frac{\partial \psi}{\partial r}.$$

(6 marks)

(ii) Show that the velocity field can be written as

$$\mathbf{u} = \mathbf{V} \frac{1}{r} \frac{df}{dr} + \mathbf{x} \frac{\mathbf{V} \cdot \mathbf{x}}{r^2} \left(\frac{2f}{r^2} - \frac{1}{r} \frac{df}{dr} \right).$$

(6 marks)

(iii) Using Cartesian coordinates, deduce that

$$\frac{\partial u_i}{\partial x_j} = -\frac{3}{4} \left(\frac{a}{r^3} + \frac{a^3}{r^5} \right) V_i x_j$$

$$+ \left(-\frac{9a}{4r^5} + \frac{15a^3}{4r^7} \right) x_i x_j V_k x_k + \frac{3}{4} \left(\frac{a}{r^3} - \frac{a^3}{r^5} \right) (V_j x_i + V_k x_k \delta_{ij}).$$

(9 marks)

(iv) Verify that $\nabla \cdot \mathbf{u} = 0$. ***(4 marks)***

End of Question Paper