The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012-2013**

Vectors and Fluids **2 hours**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1 (i) A vector field is given by $\boldsymbol{F} = (\alpha x \mathrm{e}^{2y},\, \mathrm{e}^{-3x} - \mathrm{e}^{2y},\, 0)$, where α is a constant. Calculate

$$\boldsymbol{\nabla}\cdot\boldsymbol{F}, \quad \boldsymbol{\nabla}\times\boldsymbol{F}, \quad \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{F}), \quad \boldsymbol{\nabla}^2\boldsymbol{F},$$

and verify that the following result holds for this case:

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) = \boldsymbol{\nabla}^2\boldsymbol{F} + \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{F}).$$

(9 marks)

Using your results, determine the value of α that ensures the existence of a vector $\boldsymbol{A}$ such that $\boldsymbol{F} = \boldsymbol{\nabla}\times\boldsymbol{A}$. Let $\boldsymbol{A} = A\,\boldsymbol{k}$, where $A = A(x, y)$ and $A = 0$ at the origin. Find the function A.

(8 marks)

(ii) A scalar field is given by $\phi = z - x^2 - y^2$. Sketch isosurfaces for ϕ. Calculate the rate of change of ϕ in the direction $(1, 1, 0)$ at the point $(3, 5, 0)$.

(8 marks)

2 (i) Find an expression for the i^{th}-component of $\boldsymbol{\nabla} \times (\phi\, \boldsymbol{\nabla}\psi)$, where ϕ and ψ are scalars. *Using suffix notation*, show that

$$\boldsymbol{\nabla} \times (\phi\, \boldsymbol{\nabla}\psi) = \boldsymbol{\nabla}\phi \times \boldsymbol{\nabla}\psi.$$

(6 marks)

(ii) Consider the scalar

$$\phi = \frac{\boldsymbol{\kappa} \cdot \boldsymbol{r}}{r^n},$$

where $r = |\boldsymbol{r}|$, $\boldsymbol{r} = (x_1, x_2, x_3)$ and $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \kappa_3)$ is a constant vector. Using suffix notation, show that

$$\frac{\partial \phi}{\partial x_i} = \frac{\kappa_i}{r^n} - \frac{n\,(\boldsymbol{\kappa} \cdot \boldsymbol{r})\, x_i}{r^{n+2}}.$$

(You may assume that $\partial r/\partial x_i = x_i/r$.) Find the n for which ϕ satisfies the Laplace equation.

(11 marks)

(iii) Let L be a 3×3 matrix. State the conditions on L for it to be a valid matrix of transformation, representing a rotation of frames about a common origin. Define vectors $\boldsymbol{u} = \dfrac{1}{\sqrt{2}}\boldsymbol{e}_1 + \dfrac{1}{\sqrt{2}}\boldsymbol{e}_2\,,\ \ \boldsymbol{v} = \boldsymbol{e}_3\,,$ and let

$$L = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 & 0 \\ -1 & \sqrt{3} & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

be a valid matrix of transformation. Determine which of the two perpendicular vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ points along the axis of rotation, and calculate the angle of rotation between frames.

(8 marks)

3 The spherical polar coordinates are related to the Cartesian system by

$$\alpha_1 = r = (x^2 + y^2 + z^2)^{\frac{1}{2}}, \quad \alpha_2 = \phi = \cos^{-1}(z/r), \quad \alpha_3 = \theta = \tan^{-1}(y/x),$$

or,

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

(i) Calculate $\boldsymbol{\nabla}\alpha_1$ and $\boldsymbol{\nabla}\alpha_3$. Using these results, find expressions for h_1 and h_3 (standard notation) in terms of r and θ. Note that $\mathrm{d}(\tan^{-1}\xi)/\mathrm{d}\xi = 1/(1+\xi^2)$.

Given that

$$\boldsymbol{\nabla}\alpha_2 = \frac{1}{r^2(x^2+y^2)^{\frac{1}{2}}}\,(xz\boldsymbol{i} + yz\boldsymbol{j} - (x^2+y^2)\boldsymbol{k}),$$

and $h_2 = r$, show that the α_i define a right-handed coordinate system.

(11 marks)

(ii) The vector field $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = (x^2 + y^2 + 2z^2)\,\boldsymbol{k}.$$

Sketch the volume V bounded by the surface S, defined by $x^2+y^2+z^2 = a^2$ and $z = 0$. Verify that Gauss's Theorem, namely

$$\int_V \boldsymbol{\nabla}\cdot\boldsymbol{F}\,\mathrm{d}V = \int_S \boldsymbol{F}\cdot\mathbf{d}\boldsymbol{S}\,,$$

holds for this case.

(14 marks)

4 The Euler equation for the motion $\boldsymbol{u}$ of an inviscid fluid is

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \boldsymbol{u} = -\frac{1}{\rho} \boldsymbol{\nabla} p + \boldsymbol{F},$$

where p is the pressure, ρ is the density and $\boldsymbol{F}$ is a body force.

(i) Show that

$$\frac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u} + \frac{p}{\rho} = \beta,$$

where β is a uniform constant throughout the flow, provided that the following conditions hold:

- body forces can be neglected,
- the flow does not change in time,
- the flow is irrotational,
- the density is uniform.

(7 marks)

(ii) The flow $\boldsymbol{u}$ of an incompressible fluid around a cylinder of radius a is given by

$$\boldsymbol{u} = U \left(1 - \frac{a^2}{r^2}\right) \cos\theta \, \hat{\boldsymbol{r}} - U \left[\left(1 + \frac{a^2}{r^2}\right) \sin\theta + \frac{a}{r}\right] \hat{\boldsymbol{\theta}}\,,$$

in cylindrical polar coordinates (r, θ, z), where U is a constant. Verify that the appropriate boundary conditions are satisfied on $r = a$, and find the flow far from the origin.

If p tends to the constant value p_∞ as $r \to \infty$, find an expression for the pressure on the surface of the cylinder.

(9 marks)

(iii) The flow in part (ii) has two stagnation points below the line $y = 0$. Sketch the flow and determine the lift, i.e. the force in the y direction, per unit length of the cylinder.

(9 marks)

5 The generalised gradient and curl are given by

$$\boldsymbol{\nabla}\phi = \left(\frac{1}{h_1}\frac{\partial\phi}{\partial\alpha_1}, \frac{1}{h_2}\frac{\partial\phi}{\partial\alpha_2}, \frac{1}{h_3}\frac{\partial\phi}{\partial\alpha_3}\right), \boldsymbol{\nabla}\times\mathbf{F} = \frac{1}{h_1\,h_2\,h_3}\begin{vmatrix} h_1\,\hat{\boldsymbol{\alpha}}_1 & h_2\,\hat{\boldsymbol{\alpha}}_2 & h_3\,\hat{\boldsymbol{\alpha}}_3 \\ \dfrac{\partial}{\partial\alpha_1} & \dfrac{\partial}{\partial\alpha_2} & \dfrac{\partial}{\partial\alpha_3} \\ h_1\,F_1 & h_2\,F_2 & h_3\,F_3 \end{vmatrix}.$$

The velocity potential for a flow is given by

$$\phi = \frac{\kappa}{2\pi}\theta\,,$$

everywhere except at the origin, in cylindrical polar coordinates, and where κ is a scalar constant.

(i) The streamfunction ψ for the flow is related to the velocity field $\boldsymbol{u}$ by $\boldsymbol{u} = \boldsymbol{\nabla}\times(\psi\hat{\boldsymbol{z}})$, where $\hat{\boldsymbol{z}}$ is the unit vector in the z direction. Find ψ and describe the flow in a few words.

(13 marks)

(ii) By considering a circle of radius a centred on the origin, calculate the circulation,

$$\oint_C \boldsymbol{u}\cdot\mathbf{d}\boldsymbol{l}\,.$$

(5 marks)

(iii) State Stokes' Theorem and evaluate the surface integral for this case. Explain briefly why Stokes' Theorem cannot not be applied trivially for this case.

(7 marks)

End of Question Paper